

STABILITY CONDITIONS AND QUANTUM DILOGARITHM IDENTITIES FOR DYNKIN QUIVERS

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ABSTRACT. We study fundamental group of the exchange graphs for the bounded derived category $\mathcal{D}(Q)$ of a Dynkin quiver Q and the finite-dimensional derived category $\mathcal{D}(\Gamma_N Q)$ of the Calabi-Yau- N Ginzburg algebra associated to Q . In the case of $\mathcal{D}(Q)$, we prove that its space of stability conditions (in the sense of Bridgeland) is simply connected; as applications, we show that its Donaldson-Thomas invariants can be calculated via a quantum dilogarithm function on exchange graphs. In the case of $\mathcal{D}(\Gamma_N Q)$, we show that faithfulness of the Seidel-Thomas braid group action (which is known for Q of type A or $N = 2$) implies the simply connectedness of its space of stability conditions; moreover we provide a topological realization of almost completed cluster tilting objects.

Key words: space of stability conditions, Calabi-Yau- N Ginzburg algebra, higher cluster category, Donaldson-Thomas invariant, quantum dilogarithm identity

1. INTRODUCTION

1.1. Overall. The notion of a stability condition on a triangulated category was defined by Bridgeland [6] (c.f. Section 2.8). The idea was inspired from physics by studying D-branes in string theory. Nevertheless, the notion itself is interesting purely mathematically. A stability condition on a triangulated category \mathcal{D} consists of a collection of full additive subcategories of \mathcal{D} , known as the slicing, and a group homomorphism from the Grothendieck group $\mathcal{K}(\mathcal{D})$ to the complex plane, known as the central charge. Bridgeland [6] showed a key result that the space $\text{Stab}(\mathcal{D})$ of stability conditions on \mathcal{D} is a finite dimensional complex manifold. Moreover, these spaces carry interesting geometric/topological structure which shed light on the properties of the original triangulated categories. Most interesting examples of triangulated categories are derived categories. They are weak homological invariants arising in both algebraic geometry and representation theory, and indeed different manifolds and quivers (usually with relation) might share the same derived category (say complex projective line and Kronecker quiver). Also note that the space of stability conditions are related to Kontsevich's homological mirror symmetry, that the (quotient) space of stability conditions of the Fukaya categories of Lagrangian submanifolds of certain symplectic manifolds are supposed to be some Kähler moduli space. We will study the spaces of stability conditions of the bounded derived category $\mathcal{D}(Q)$ of a Dynkin quiver Q and the finite-dimensional derived category $\mathcal{D}(\Gamma_N Q)$ of the Calabi-Yau- N Ginzburg algebra associated to Q . Noticing that when Q is of Dynkin type, $\mathcal{D}(\Gamma_N Q)$ was studied by Khovanov-Seidel-Thomas [27]/[37]

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via the derived Fukaya category of Lagrangian submanifolds of the Milnor fibres of the singularities of type A_n .

In understanding stability conditions and triangulated categories, t-structures play an important role. In fact, we can view a t-structure as a ‘discrete’ (integer) structure while a stability condition (resp. a slicing) is its ‘complex’ (resp. ‘real’) refinement. Every t-structure carries an abelian category sitting inside it, known as its heart. Note that an abelian category is a canonical heart in its derived category, e.g. $\mathcal{H}_Q = \text{mod } \mathbf{k}Q$ is the canonical heart of $\mathcal{D}(Q)$. The classical way to understand relations between different hearts is via HRS-tilting (c.f. Section 2.7), in the sense of Happel-Reiten-Smalø. To give a stability condition is equivalent to giving a t-structure and a stability function on its heart with the Harder-Narashimhan (HN) property. This implies that a finite heart (i.e. has n simples and has finite length) corresponds to a (complex) n -cell in the space of stability conditions. Moreover, Woolf [41] shows that the tilting between finite hearts corresponds to the tiling of these n -cells. More precisely, two n -cells meet if and only if the corresponding hearts differ by a HRS-tilting; and they meet in codimension one if and only if the hearts differ by a simple tilting. Thus, our main method to study a space of stability conditions of a triangulated category \mathcal{D} is via its ‘skeleton’ – the exchange graph $\text{EG}(\mathcal{D})$, that is, the oriented graphs whose vertices are hearts in \mathcal{D} and whose edges correspond to simple (forward) tiltings between them (c.f. [35]). Figure 1 (taken from [35], which in fact, the quotient graph of $\text{EG}^\circ(\mathcal{D}(\Gamma_3 A_2))/[1]$ and $\text{Stab}^\circ(\mathcal{D}(\Gamma_3 A_2))/\mathbb{C}$) demonstrates the duality between the exchange graph and the tiling of the space of stability conditions by many cells like the shaded area, so that each vertex in the exchange graph corresponds to a cell and each edge corresponds to a common edge (codimension one face) of two neighboring cells. We will prove certain simply connectedness of spaces of stability conditions via exchange graph.

Stability conditions naturally link to Donaldson-Thomas (DT) invariants, which was originally defined as the weighted Euler characteristics (By Behrend function) of moduli spaces for Calabi-Yau 3-folds (c.f. [32]). Reineke [38] (c.f. Section 7.1) realized that the DT-invariant for a Dynkin quiver can be calculated as a product of quantum dilogarithms, indexing by any HN-stratum of \mathcal{H}_Q , which is a ‘maximal refined version’ of torsion pairs on an abelian category. His approach was integrating certain identities in Hall algebras to show the stratum-independence of the product. We will apply exchange graphs to give a combinatorial proof of such quantum dilogarithm identities.

1.2. Contents. We will collect related background in Section 2.

In Section 3 and Section 4, we first make a key observation (Proposition 3.5) that the fundamental group of the exchange graphs generates by squares and pentagons. Then we prove (Theorem 3.7) the simply connectedness of the space of stability conditions $\text{Stab}(\mathcal{D}(Q))$ and show that (Corollary 4.6) the faithfulness of the Seidel-Thomas braid group action (which is known for Q of type A or $N = 2$) implies the simply connectedness of its space of stability conditions. Moreover, the quotient space of $\text{Stab}^\circ(\Gamma_N Q)$ by the Seidel-Thomas braid group $\text{Br}(\Gamma_N Q)$ is the ‘right’ space of stability conditions for the higher cluster category $\mathcal{C}_{N-1}(Q)$ (see Remark 4.7). In fact, the generators of its fundamental group provide a topological realization of almost completed cluster tilting objects in $\mathcal{C}_{N-1}(Q)$ (Theorem 4.5).

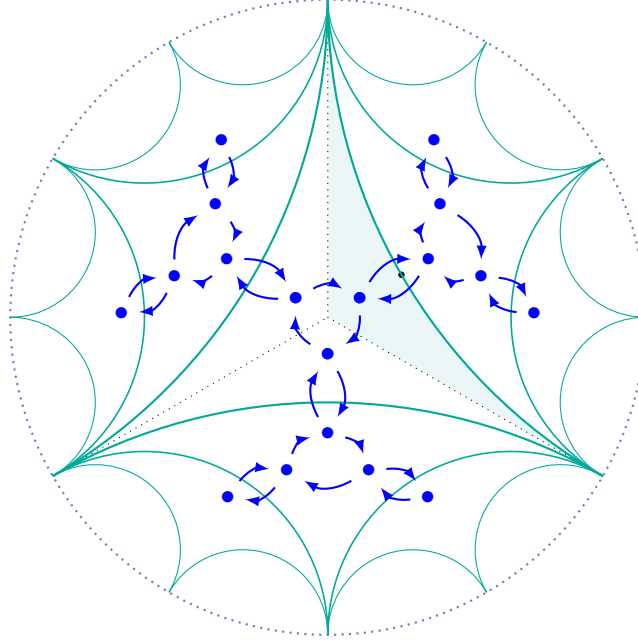


FIGURE 1. Exchange graphs as the skeleton of space of stability conditions

In Section 5, we present (Theorem 5.2) a limit formula of spaces of stability conditions

$$\text{Stab}(Q) \cong \lim_{N \rightarrow \infty} \text{Stab}^\circ(\Gamma_N Q) / \text{Br}(\Gamma_N Q),$$

which reflects a philosophical point of view that, in a suitable sense,

$$Q = \lim_{N \rightarrow \infty} \Gamma_N Q. \quad (1.1)$$

In Section 6, we study directed paths in exchange graphs. We will first show (Theorem 6.9) that HN-strata of \mathcal{H}_Q can be naturally interpreted as directed paths connecting \mathcal{H}_Q and $\mathcal{H}_Q[1]$ in $\text{EG}(Q)$. Then we discuss total stability of stability functions (c.f. Conjecture 6.13) and the path-inducing problem. We will provide explicit examples and a conjecture.

In Section 7, we observe that the existence of DT-invariant of Q is equivalent to the path-independence of the quantum dilogarithm product over certain directed paths. Then we give a slight generalization (Theorem 7.3) of this path-independence, to all paths (not necessarily directed) whose vertices lie between \mathcal{H}_Q and $\mathcal{H}_Q[1]$. The point is that this path-independence reduces to the cases of squares and pentagons in Proposition 3.5; therefore such type of quantum dilogarithm identities are just compositions of the classical Pentagon Identities. We will also discuss the wall-crossing formula for APR-tilting (c.f. [31]). Note that Keller [20] also spotted this phenomenon and proved a more remarkable quantum dilogarithm identities via mutation of quivers with potential. In fact, his formula can also be rephrased as quantum dilogarithm product over paths in the exchange graph of the corresponding Calabi-Yau-3 categories.

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2. PRELIMINARIES

2.1. Dynkin Quivers. A (*simply laced*) *Dynkin quiver* $Q = (Q_0, Q_1)$ is a quiver whose underlying unoriented graph is one of the following unoriented graphs:

$$\begin{aligned}
 A_n : & \quad 1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n \\
 D_n : & \quad 1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n-2 \begin{array}{l} \nearrow n-1 \\ \searrow n \end{array} \\
 E_{6,7,8} : & \quad \begin{array}{ccccccc} & & 4 & & & & \\ & & | & & & & \\ 1 & \text{---} & 2 & \text{---} & 3 & \text{---} & 5 \text{ --- } 6 \text{ --- } 7 \text{ --- } 8 \end{array}
 \end{aligned} \tag{2.1}$$

For a Dynkin quiver Q , we denote by $\mathbf{k}Q$ the *path algebra*; denote by $\text{mod } \mathbf{k}Q$ the *category of finite dimensional $\mathbf{k}Q$ -modules*, which can be identified with $\text{Rep}_{\mathbf{k}}(Q)$, the *category of representations* of Q (c.f. [2]). We will not distinguish between $\text{mod } \mathbf{k}Q$ and $\text{Rep}_{\mathbf{k}}(Q)$. Recall that the *Euler form*

$$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$$

associated to the quiver Q , is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{(i \rightarrow j) \in Q_1} \alpha_i \beta_j.$$

Moreover for $M, L \in \text{mod } \mathbf{k}Q$, we have

$$\langle \dim M, \dim L \rangle = \dim \text{Hom}(M, L) - \dim \text{Ext}^1(M, L), \tag{2.2}$$

where $\dim E \in \mathbb{N}^{Q_0}$ is the *dimension vector* of any $E \in \text{mod } \mathbf{k}Q$.

2.2. Hearts and t-structures. Let $\mathcal{D}(Q) = \mathcal{D}^b(\text{mod } \mathbf{k}Q)$ be the *bounded derived category* of Q , which is a triangulated category.

Recall (e.g. from [6]) that a *t-structure* on a triangulated category \mathcal{D} is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ and such that, if one defines

$$\mathcal{P}^\perp = \{G \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(F, G) = 0, \forall F \in \mathcal{P}\},$$

then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \rightarrow E \rightarrow G \rightarrow F[1]$ in \mathcal{D} with $F \in \mathcal{P}$ and $G \in \mathcal{P}^\perp$. Any t-structure is closed under sums and summands and hence it is determined by the indecomposables in it. Note also that $\mathcal{P}^\perp[-1] \subset \mathcal{P}^\perp$. A t-structure \mathcal{P} is *bounded* if for every object M , the shifts $M[k]$ are in \mathcal{P} for $k \gg 0$ and in \mathcal{P}^\perp for $k \ll 0$. The *heart* of a t-structure \mathcal{P} is the full subcategory

$$\mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P}$$

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \dots \xrightarrow{f_1} M_{t-1} \xrightarrow{f_t} M_t$$

of irreducible maps f_i between indecomposable modules M_i with $t \geq 1$. When such a path exists, we say that M_0 is a *predecessor* of M_t or M_t is a *successor* of M_0 .

- A path $M_0 \rightarrow \dots \rightarrow M_t$ in $\Lambda(\mathcal{C})$ is called *sectional* if, for all $1 < i \leq t$, $\tau M_i \not\cong M_{i-2}$.
- Let $\text{Ps}(M)$ be the set of objects that lie in some sectional path starting from M and $\text{Ps}^{-1}(M)$ be the set of objects that lie in some sectional path ending at M .

We have the following elementary lemma.

Lemma 2.1 ([2]). *We have*

- 1°. *Any section in $\mathbb{Z}Q$ is isomorphic to some orientation of Δ .*
- 2°. *For any object M in $\mathbb{Z}Q$, $\text{Ps}(M)$ and $\text{Ps}^{-1}(M)$ are sections.*
- 3°. *The projectives of \mathcal{H}_Q together with the irreducible maps between them are a section in $\Lambda(\mathcal{D}(Q))$. Moreover the section has the exactly the opposite orientation of Q .*

For a section P in $\Lambda(\mathcal{D}(Q)) \cong \mathbb{Z}Q$, define $[P, \infty) = \bigcup_{m \geq 0} \tau^{-m}P$. Similarly for $(-\infty, P]$ and define

$$[P_1, P_2] = [P_1, \infty) \cap (-\infty, P_2].$$

The following lemmas characterize such type of intervals.

Lemma 2.2. *The interval $[\text{Ps}(M), \infty)$ consists precisely of all the successors of M . Similarly, $(-\infty, \text{Ps}^{-1}(M)]$ consists precisely all the predecessors of M .*

Proof. We only prove the first assertion. The second is similar.

By the local property of the translation quiver $\mathbb{Z}Q$, any object in $[\text{Ps}(M), \infty)$ is a successor of M . On the other hand, let L be any successor of M with path

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_j} M_j = L.$$

If $\tau M_i = M_{i-2}$ for some $3 \leq i \leq j$, then consider τL with path

$$M = M_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-2}} M_{i-2} = \tau M_i \xrightarrow{\tau f_i} \tau M_{i+1} \xrightarrow{\tau f_{i+1}} \dots \xrightarrow{\tau f_j} \tau M_j = \tau L.$$

we can repeat the process until the path is sectional, i.e. until we obtain $\tau^k L \in \text{Ps}(M)$ for some $k \geq 0$. Thus $L \in [\text{Ps}(M), \infty)$. \square

Lemma 2.3. *Let $M, L \in \text{Ind } \mathcal{D}(Q)$. If $\text{Hom}(M, L) \neq 0$ then*

$$L \in [\text{Ps}(M), \text{Ps}^{-1}(\tau(M[1]))], \quad M \in [\text{Ps}(\tau^{-1}(L[-1])), \text{Ps}^{-1}(L)].$$

Proof. By the Auslander-Reiten formula, we have

$$\text{Hom}(M, L)^* = \text{Hom}(\tau^{-1}(L), M[1]).$$

The lemma now follows from Lemma 2.2. \square

For later use, we define the position function as follows.

Definition/Lemma 2.4. There is a *position function* $\text{pf} : \Lambda(\mathcal{D}(Q)) \rightarrow \mathbb{Z}$, unique up to an additive constant, such that $\text{pf}(M) - \text{pf}(\tau M) = 2$ for any $M \in \Lambda(\mathcal{D}(Q))$ and $\text{pf}(M) < \text{pf}(L)$ for any successor L of M . For a heart \mathcal{H} in $\mathcal{D}(Q)$, define

$$\text{pf}(\mathcal{H}) = \sum_{S \in \text{Sim } \mathcal{H}} \text{pf}(S).$$

2.4. Standard hearts in $\mathcal{D}(Q)$.

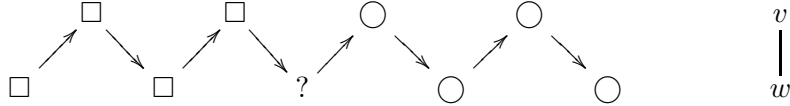
Proposition 2.5. A section P in $\mathcal{D}(Q)$ will induce a unique t-structure \mathcal{P} on $\mathcal{D}(Q)$ such that $\text{Ind } \mathcal{P} = [P, \infty)$. For any t-structure \mathcal{P} on $\mathcal{D}(Q)$, the followings are equivalent

- 1°. \mathcal{P} is induced by some section P .
- 2°. $\text{Ind } \mathcal{D}(Q) = \text{Ind } \mathcal{P} \cup \text{Ind } \mathcal{P}^\perp$.
- 3°. The corresponding heart \mathcal{H} is isomorphic to \mathcal{H}'_Q , where Q' has the same underlying diagram of Q .
- 4°. $\text{Wid}_{\mathcal{H}} M = 0$ for any $M \in \text{Ind } \mathcal{D}(Q)$, where \mathcal{H} is the corresponding heart.

Proof. For a section P , let \mathcal{P} be the subcategory which is generated by the elements in $\text{Ind } \mathcal{P} = [P, \infty)$. Notice that $\text{Ind } \mathcal{P}^\perp = (\infty, \tau^{-1}P]$ which implies \mathcal{P} is a t-structure. Thus $1^\circ \Rightarrow 2^\circ$. Since $\mathcal{H} = [P, P[1])$, $1^\circ \Rightarrow 3^\circ$.

If \mathcal{H} is isomorphic to \mathcal{H}'_Q for some quiver Q' , then $\text{Ind } \mathcal{P} = \cup_{j \geq 0} \mathcal{H}[j] = [P', \infty)$, where P' is the sub-quiver in $\Lambda(\mathcal{D}(Q))$ consists of the projectives. Thus $3^\circ \Rightarrow 1^\circ$. Since for any $M \in \text{Ind } \mathcal{D}(Q)$, $\text{Wid}_{\mathcal{H}} M = 0$ if and only if $M \in \mathcal{H}[k]$ for some integer k , we have $3^\circ \Rightarrow 4^\circ$. Noticing that $\mathcal{H}[k]$ is either in \mathcal{P} or \mathcal{P}^\perp , we have $4^\circ \Rightarrow 2^\circ$.

Now we only need to prove $2^\circ \Rightarrow 1^\circ$. If an indecomposable M is in \mathcal{P} (resp. \mathcal{P}^\perp), then, inductively, all of its successors (resp. predecessors) are in \mathcal{P} (resp. \mathcal{P}^\perp). By the local property, $\tau^m(M)$ is a successor of M if $m \geq 0$ and a predecessor if $m \leq 0$. Hence, in any row $\pi^{-1}(v) \in \mathbb{Z}Q \cong \Lambda(\mathcal{D}(Q))$, for any vertex $v \in Q_0$, there is a unique integer m_v such that $\tau^j(v) \in \mathcal{P}$, for $j \geq m_v$, while $\tau^j(v) \in \mathcal{P}^\perp$, for $j < m_v$. Furthermore, for a neighboring vertex w of v , the local picture looks like this



where $\bigcirc \in \mathcal{P}$ and $\square \in \mathcal{P}^\perp$. Hence v_{m_v} and w_{m_w} must be connected in $\mathbb{Z}Q$ and so the full sub-quiver of $\mathbb{Z}Q$ consisting of all vertices $\{v_{m_v}\}_{v \in Q_0}$ is a section and furthermore it induces \mathcal{P} . \square

We call a heart on $\mathcal{D}(Q)$ is *standard* if the corresponding t-structure is induced by a section.

2.5. Calabi-Yau categories. Let $N > 1$ be an integer. Denote by $\Gamma_N Q$ the *Calabi-Yau- N Ginzburg (dg) algebra* associated to Q , that is, the dg algebra

$$\mathbf{k}Q_0 \langle x, x^*, e^* \mid x \in Q_1, e \in Q_0 \rangle$$

with degrees

$$\deg e = \deg x = 0, \quad \deg x^* = N - 2, \quad \deg e^* = N - 1$$

and only nontrivial differentials

$$d \sum_{e \in Q_0} e^* = \sum_{x \in Q_1} [x, x^*].$$

Write $\mathcal{D}(\Gamma_N Q)$ for $\mathcal{D}_{fd}(\text{mod } \Gamma_N Q)$.

Recall that a triangulated category \mathcal{C} is called *Calabi-Yau- N* if, for any objects L, M in \mathcal{C} we have a natural isomorphism

$$\mathfrak{S} : \text{Hom}_{\mathcal{C}}^{\bullet}(L, M) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}^{\bullet}(M, L)^{\vee}[N]. \quad (2.6)$$

An object S is *N -spherical* if $\text{Hom}^{\bullet}(S, S) = \mathbf{k} \oplus \mathbf{k}[-N]$.

By [24] (see also [27],[37],[40]), we know that $\mathcal{D}(\Gamma_N Q)$ is a Calabi-Yau- N category which admits a standard heart \mathcal{H}_{Γ} generated by simple $\Gamma_N Q$ -modules $S_e, e \in Q_0$, each of which is N -spherical. Denote by $\text{EG}^{\circ}(\Gamma_N Q)$ the principal component of the exchange graph $\text{EG}(\mathcal{D}(\Gamma_N Q))$, that is, the component containing \mathcal{H}_{Γ} .

2.6. Twist functors and braid groups. We recall (c.f. [27],[37],[40]) a distinguished family of auto-equivalences in $\text{Aut } \mathcal{D}(\Gamma_N Q)$, for the CY- N category $\mathcal{D}(\Gamma_N Q)$.

Definition 2.6. The *twist functor* ϕ of a spherical object S is defined by

$$\phi_S(X) = \text{Cone}(S \otimes \text{Hom}^{\bullet}(S, X) \rightarrow X). \quad (2.7)$$

with inverse

$$\phi_S^{-1}(X) = \text{Cone}(X \rightarrow S \otimes \text{Hom}^{\bullet}(X, S)^{\vee})[-1] \quad (2.8)$$

The *Seidel-Thomas braid group*, denoted by $\text{Br}(\Gamma_N Q)$, is the subgroup of $\text{Aut } \mathcal{D}(\Gamma_N Q)$ generating by the twist functors of the simples in $\text{Sim } \mathcal{H}_{\Gamma}$.

2.7. Exchange graphs. A similar notion to a t-structure on a triangulated category is a torsion pair on an abelian category. Tilting with respect to a torsion pair in the heart of a t-structure provides a way to pass between different t-structures.

Definition 2.7. A *torsion pair* in an abelian category \mathcal{C} is a pair of full subcategories $\langle \mathcal{F}, \mathcal{T} \rangle$ of \mathcal{C} , such that $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and furthermore every object $E \in \mathcal{C}$ fits into a short exact sequence $0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0$ for some objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

We will use the notation $\mathcal{H} = \langle \mathcal{F}, \mathcal{T} \rangle$ to indicate an abelian category with a torsion pair.

Proposition 2.8 (Happel, Reiten, Smalø). *Let $\mathcal{H} = \langle \mathcal{F}, \mathcal{T} \rangle$ be a heart in a triangulated category \mathcal{D} . Then there exists the following two hearts with torsion pairs*

$$\mathcal{H}^{\sharp} = \langle \mathcal{T}, \mathcal{F}[1] \rangle, \quad \mathcal{H}^{\flat} = \langle \mathcal{T}[-1], \mathcal{F} \rangle.$$

We call \mathcal{H}^{\sharp} the *forward tilt* of \mathcal{H} with respect to the torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$, and \mathcal{H}^{\flat} the *backward tilt* of \mathcal{H} . Clearly $\mathcal{H}^{\flat} = \mathcal{H}^{\sharp}[-1]$.

For two hearts $\mathcal{H}_i, i = 1, 2$ in \mathcal{D} with corresponding t-structure \mathcal{P}_i , we say $\mathcal{H}_1 \leq \mathcal{H}_2$ if and only if $\mathcal{P}_1 \supset \mathcal{P}_2$, or equivalently, $\mathcal{P}_1^{\perp} \subset \mathcal{P}_2^{\perp}$, with equality if and only if the two hearts are the same.

The following lemma collect several well-known facts about tilting.

Lemma 2.9 ([16],[35]). *In any triangulated category, we have*

- $\mathcal{H}[-1] \leq \mathcal{H}^b \leq \mathcal{H} \leq \mathcal{H}^\sharp \leq \mathcal{H}[1]$.
- $\mathcal{H} \leq \mathcal{H} \leq \mathcal{H}[1]$ if and only if $\mathcal{H}' = \mathcal{H}^\sharp$. Moreover, in such case, the tilting is with respect to the torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$ in \mathcal{H} , where $\mathcal{T} = \mathcal{H} \cap \mathcal{H}'$ and $\mathcal{F} = \mathcal{H} \cap \mathcal{H}'[-1]$.

We say a forward tilting is *simple*, if the corresponding torsion free part is generated by a single simple object S , and denote the heart by \mathcal{H}_S^\sharp . Similarly, a backward tilting is *simple* if the corresponding torsion part is generated by a single simple object S , and denote the heart by \mathcal{H}_S^b .

Definition 2.10. Define the *exchange graph* $\text{EG}(\mathcal{D})$ of a triangulated category \mathcal{D} to be the oriented graph whose vertices are all hearts in \mathcal{D} and whose edges correspond to simple forward tiltings between them.

We will label an edge of $\text{EG}(\mathcal{D})$ by the simple object of the corresponding tilting, i.e. the edge with end points \mathcal{H} and \mathcal{H}_S^\sharp will be labeled by S . For $S \in \text{Sim } \mathcal{H}$, inductively define

$$\mathcal{H}_S^{m\sharp} = \left(\mathcal{H}_S^{(m-1)\sharp} \right)_{S[m-1]}^\sharp$$

for $m \geq 1$ and similarly for \mathcal{H}_S^{mb} , $m \geq 1$. We will write $\mathcal{H}_S^{m\sharp} = \mathcal{H}_S^{-mb}$ for $m < 0$.

Definition 2.11. A *line* $l = l(\mathcal{H}, S)$ in $\text{EG}(\mathcal{D})$, for some triangulated category \mathcal{D} , is the full subgraph consisting of the vertices $\{\mathcal{H}_S^{m\sharp}\}_{m \in \mathbb{Z}}$, for some heart \mathcal{H} and a simple $S \in \text{Sim } \mathcal{H}$. We say an edge in $\text{EG}(\mathcal{D})$ has *direction* T if its label is $T[m]$ for some integer m ; we say a line l has *direction- T* if some (and hence every) edge in l has direction T .

By [25], we know that $\text{EG}(\mathcal{D}(Q))$ is connected when Q is of Dynkin type, which will be wrote as $\text{EG}(Q)$. For an alternate proof, see Appendix A. Denote by $\text{EG}^\circ(\Gamma_N Q)$ the principal component of the exchange graph $\text{EG}(\mathcal{D}(\Gamma_N Q))$, that is, the component containing \mathcal{H}_Γ .

Recall some notation and results from [35]. There is special kind of exact functors from $\mathcal{D}(Q)$ to $\mathcal{D}(\Gamma_N Q)$, known as the *Lagrangian immersions* (L-immersions), see [35, Definition 6.2]. Let \mathcal{H} be a heart in $\mathcal{D}(\Gamma_N Q)$ with $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$. If there is a L-immersion $F : \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma_N Q)$ and a heart $\widehat{\mathcal{H}} \in \text{EG}^\circ(Q)$ with $\text{Sim } \widehat{\mathcal{H}} = \{\widehat{S}_1, \dots, \widehat{S}_n\}$, such that $F(\widehat{S}_i) = S_i$, then we say that \mathcal{H} is induced via F from $\widehat{\mathcal{H}}$ and write $F_*(\widehat{\mathcal{H}}) = \mathcal{H}$.

Further, let \mathcal{H} be a heart in some exchange graph $\text{EG}^\circ(Q)$. Define the exchange graph $\text{EG}_N(Q, \mathcal{H})$ with base \mathcal{H} to be the full subgraph of $\text{EG}(Q)$ induced by

$$\{\mathcal{H}_0 \mid \mathcal{H} \in \text{EG}(Q), \mathcal{H}[1] \leq \mathcal{H}_0 \leq \mathcal{H}[N-1]\}$$

and $\text{EG}_N^\circ(Q, \mathcal{H})$ its principal component (that is, the connected component containing $\mathcal{H}[1]$). Similarly for $\text{EG}_N(\Gamma_N Q, \mathcal{H})$ and $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H})$.

Theorem 2.12 ([35]). *For an acyclic quiver Q , we have the following:*

- 1°. *there is a canonical L-immersion $\mathcal{I} : \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma_N Q)$ that induces an isomorphism*

$$\mathcal{I}_* : \text{EG}_N^\circ(Q, \mathcal{H}_Q) \rightarrow \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma). \quad (2.9)$$

- 2°. *as vertex set, we have*

$$\text{EG}(Q, \mathcal{H}_Q) \cong \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) \cong \text{EG}^\circ(\Gamma_N Q) / \text{Br}; \quad (2.10)$$

- 3°. for any heart \mathcal{H} in $\text{EG}^\circ(\Gamma_N Q)$, $\text{Sim } \mathcal{H}$ has n elements and $\{\phi_S\}_{S \in \text{Sim } \mathcal{H}}$ is a generating set for $\text{Br}(\Gamma_N Q)$;
- 4°. for any line l in $\text{EG}^\circ(\Gamma_N Q)$, its orbit in $\text{EG}^\circ(\Gamma_N Q)/\text{Br}$ is a $(N-1)$ cycle.

Besides, we have

Proposition 2.13. *Let Q be a Dynkin quiver. $\text{EG}_N(Q, \mathcal{H}_Q)$ is finite for any $N > 1$ and we have*

$$\text{EG}(Q) = \lim_{N \rightarrow \infty} \text{EG}_{2N}(Q, \mathcal{H}_Q[-N]). \quad (2.11)$$

Proof. Notice that there are only finitely many indecomposables in $\bigcup_{j=1}^{N-1} \mathcal{H}_Q[j]$ and hence only finitely many hearts in $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$.

Let $\mathcal{H} \in \text{EG}(Q)$. Consider the homology \mathbf{H}_\bullet , with respect to \mathcal{H}_Q , of any simple S of \mathcal{H} . Then we know that if $N \gg 1$, then $S \in \bigcup_{j=1-N}^{N-1} \mathcal{H}_Q[j]$ which implies $\mathcal{H}_Q[-N+1] \leq \mathcal{H} \leq \mathcal{H}_Q[N-1]$. Then $\mathcal{H} \in \text{EG}_{2N}(Q, \mathcal{H}_Q[-N])$ which implies (2.11). \square

2.8. Stability conditions. This section (following [6]) collects the basic definitions of stability conditions. Denote \mathcal{D} a triangulated category and $K(\mathcal{D})$ its Grothendieck group.

Definition 2.14 ([7] Definition 3.1). A *stability condition* $\sigma = (Z, \mathcal{P})$ on \mathcal{D} consists of a group homomorphism $Z : K(\mathcal{D}) \rightarrow \mathbb{C}$ called the *central charge* and full additive subcategories $\mathcal{P}(\varphi) \subset \mathcal{D}$ for each $\varphi \in \mathbb{R}$, satisfying the following axioms:

- 1°. if $0 \neq E \in \mathcal{P}(\varphi)$ then $Z(E) = m(E) \exp(\varphi \pi i)$ for some $m(E) \in \mathbb{R}_{>0}$,
- 2°. for all $\varphi \in \mathbb{R}$, $\mathcal{P}(\varphi+1) = \mathcal{P}(\varphi)[1]$,
- 3°. if $\varphi_1 > \varphi_2$ and $A_i \in \mathcal{P}(\varphi_i)$ then $\text{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,
- 4°. for each nonzero object $E \in \mathcal{D}$ there is a finite sequence of real numbers

$$\varphi_1 > \varphi_2 > \dots > \varphi_m$$

and a collection of triangles

$$0 = E_0 \xrightarrow{\quad} E_1 \xrightarrow{\quad} E_2 \xrightarrow{\quad} \dots \xrightarrow{\quad} E_{m-1} \xrightarrow{\quad} E_m = E,$$

$\swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow$
 $A_1 \quad A_2 \quad \quad \quad A_m$

with $A_j \in \mathcal{P}(\varphi_j)$ for all j .

We call the collection of subcategories $\{\mathcal{P}(\varphi)\}$, satisfying 2°-4° in Definition 2.14, the *slicing* and the collection of triangles in 4° the *Harder-Narashimhan (HN) filtration*. For any nonzero object $E \in \mathcal{D}$ with HN-filtration above, define its upper phase to be $\Psi_{\mathcal{P}}^+(E) = \varphi_1$ and lower phase to be $\Psi_{\mathcal{P}}^-(E) = \varphi_n$. By [7, Lemma 5.2], $\mathcal{P}(\varphi)$ is abelian. An object $E \in \mathcal{P}(\varphi)$ for some $\varphi \in \mathbb{R}$ is said to be semistable; in which case, $\varphi = \Psi_{\mathcal{P}}^\pm(E)$. Moreover, if E is simple in $\mathcal{P}(\varphi)$, then it is said to be stable. Let I be an interval in \mathbb{R} and define

$$\mathcal{P}(I) = \{E \in \mathcal{D} \mid \Psi_{\mathcal{P}}^\pm(E) \in I\}.$$

Then for any $\varphi \in \mathbb{R}$, $\mathcal{P}([\varphi, \infty))$ and $\mathcal{P}((\varphi, \infty))$ are t-structures in \mathcal{D} .

There is a natural \mathbb{C} action on the set $\text{Stab}(\mathcal{D})$ of all stability conditions on \mathcal{D} , namely:

$$\Theta \cdot (Z, \mathcal{P}) = (Z \cdot z, \mathcal{P}_x),$$

where $z = \exp(\Theta\pi\mathbf{i})$, $\Theta = x + y\mathbf{i}$ and $\mathcal{P}_x(m) = \mathcal{P}(x + m)$ for $x, y, m \in \mathbb{R}$. There is also a natural action on $\text{Stab}(\mathcal{D})$ induced by $\text{Aut}(\mathcal{D})$, namely:

$$\xi \circ (Z, \mathcal{P}) = (Z \circ \xi, \xi \circ \mathcal{P}).$$

Similarly to stability condition on triangulated categories, we have the notation of stability function on abelian categories.

Definition 2.15 ([6]). A *stability function* on an abelian category \mathcal{C} is a group homomorphism $Z : \mathcal{K}(\mathcal{C}) \rightarrow \mathbb{C}$ such that for any object $0 \neq M \in \mathcal{C}$, we have $Z(M) = m(M) \exp(\mu_Z(M)\mathbf{i}\pi)$ for some $m(M) \in \mathbb{R}_{>0}$ and $\mu_Z(M) \in [0, 1)$, i.e. $Z(M)$ lies in the upper half-plane

$$H = \{r \exp(\mathbf{i}\pi\theta) \mid r \in \mathbb{R}_{>0}, 0 \leq \theta < 1\} \subset \mathbb{C}. \quad (2.12)$$

Call $\mu_Z(M)$ the phase of M . We say an object $0 \neq M \in \mathcal{C}$ is semistable (with respect to Z) if every subobject $0 \neq L$ of M satisfies $\mu_Z(L) \leq \mu_Z(M)$. Further, we say a stability function Z on \mathcal{C} satisfies HN-property, if for an object $0 \neq M \in \mathcal{C}$, there is a collection of short exact sequences

$$\begin{array}{ccccccc} 0 = M_0 & \hookrightarrow & M_1 & \hookrightarrow & \dots & \hookrightarrow & M_{m-1} & \hookrightarrow & M_k = M \\ & & \searrow & & & & \searrow & & \\ & & L_1 & & & & L_k & & \end{array}$$

in \mathcal{C} such that L_1, \dots, L_k are semistable objects (with respect to Z) and their phases are in decreasing order, i.e. $\phi(L_1) > \dots > \phi(L_k)$.

Note that we have a different convention $0 \leq \theta < 1$ for the upper half plane H in (2.12) as Bridgeland's $0 < \theta \leq 1$.

Then we have another way to give a stability condition on triangulated categories.

Proposition 2.16 ([6], [7]). *To give a stability condition on a triangulated category \mathcal{D} is equivalent to giving a bounded t -structure on \mathcal{D} and a stability function on its heart with the HN-property. Further, to give a stability condition on \mathcal{D} with a finite heart \mathcal{H} is equivalent to giving a function $\text{Sim } \mathcal{H} \rightarrow H$, where H is the upper half plane as in (2.12).*

Recall a crucial result of spaces of stability conditions.

Theorem 2.17 (Bridgeland [6]). *The space of stability conditions on a triangulated category \mathcal{D} is a complex manifold, denoted by $\text{Stab}(\mathcal{D})$.*

Therefore every finite heart \mathcal{H} corresponds to a (complex, half closed and half open) n -cell $U(\mathcal{H}) \simeq H^n$ inside $\text{Stab}(\mathcal{D})$.

3. SIMPLY CONNECTEDNESS OF $\text{STAB}(\mathcal{Q})$

Let $\text{Stab}(\mathcal{Q}) = \text{Stab}(\mathcal{D}(\mathcal{Q}))$. The connectedness of $\text{Stab}(\mathcal{Q})$ follows from the connectedness of $\text{EG}(\mathcal{Q})$.

3.1. A canonical embedding. By connectedness of $\text{EG}(Q)$, we have a disjoint union $\text{Stab}(Q) = \bigcup_{\mathcal{H} \in \text{EG}(Q)} \text{U}(\mathcal{H})$. Moreover, by the results in [41, Section 2], we have

$$\overline{\text{U}(\mathcal{H})} - \text{U}(\mathcal{H}) = \bigcup_{\mathcal{H}[-1] \leq \mathcal{H}' < \mathcal{H}} \left(\overline{\text{U}(\mathcal{H})} \cap \text{U}(\mathcal{H}') \right), \quad (3.1)$$

and hence the gluing structure of $\text{Stab}(Q) = \bigcup_{\mathcal{H} \in \text{EG}(Q)} \overline{\text{U}(\mathcal{H})}$ is encoded by the following formula

$$\partial \text{U}(\mathcal{H}) = \bigcup_{\mathcal{H}[-1] \leq \mathcal{H}' < \mathcal{H}} \left(\overline{\text{U}(\mathcal{H})} \cap \text{U}(\mathcal{H}') \right) \cup \bigcup_{\mathcal{H} < \mathcal{H}' \leq \mathcal{H}[1]} \left(\overline{\text{U}(\mathcal{H}')} \cap \text{U}(\mathcal{H}) \right), \quad (3.2)$$

Call a term in the RHS in (3.2) a face of the n -cell $\text{U}(\mathcal{H})$. Further, by [7, Lemma 5.5], codimension one faces of \mathcal{H} corresponds to its simple tilts. More precisely, $\dim \overline{\text{U}(\mathcal{H})} \cap \text{U}(\mathcal{H}) = n - 1$ if and only if $\mathcal{H}' = \mathcal{H}_S^\sharp$ or $\mathcal{H}' = \mathcal{H}_S^\flat$ for some $S \in \text{Sim } \mathcal{H}$. Therefore, we have the following lemma.

Lemma 3.1. *There is a canonical embedding (unique up to homotopy)*

$$\iota : \text{EG}(Q) \hookrightarrow \text{Stab}(Q) \quad (3.3)$$

such that

- 1°. for each vertex (heart) \mathcal{H} , its image is the center of the n -cell $\text{U}(\mathcal{H})$, i.e. $\iota(\mathcal{H}) = (Z_{\mathcal{H}}, \mathcal{P}_{\mathcal{H}})$ with heart \mathcal{H} satisfying $Z_{\mathcal{H}}(S_j) = \exp(\frac{1}{2}\pi \mathbf{i})$.
- 2°. for each edge $S_i : \mathcal{H} \rightarrow \mathcal{H}_{S_i}^\sharp$, its image $\sigma_{(0,1)} = \{\sigma_t = (Z_t, \mathcal{P}_t) \mid t \in (0,1)\}$ is contained in $(\text{U}(\mathcal{H}) \cup \text{U}(\mathcal{H}_{S_i}^\sharp))^\circ$ and intersects $(\text{U}(\mathcal{H}) \cap \overline{\text{U}(\mathcal{H}_{S_i}^\sharp)})^\circ$ exactly once.

Now we fix a canonical embedding ι and will identify the exchange graph with the image of this embedding.

Lemma 3.2. *We have a surjection $\pi_1(\text{EG}(Q)) \rightarrow \pi_1(\text{Stab}(Q))$.*

Proof. Let Y be the union of all faces, with codimension bigger than one, of some heart in $\text{EG}(Q)$. We can slightly perturb any path in $\text{Stab}(Q)$, without changing its class in $\pi_1(\text{Stab}(Q))$, such that it misses Y . Since $\text{Stab}^\circ(Q) - Y$ contracts onto $\text{EG}(Q)$, the lemma follows. \square

3.2. Simply connectedness. First, we prove two elementary but important lemmas.

Lemma 3.3. *Let \mathcal{H} be a heart of $\mathcal{D}(Q)$ with $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$ and $\mathcal{E}_{ij} = \text{Hom}^\bullet(S_i, S_j)$. Then for $i \neq j, j \neq k$,*

- 1°. $\dim \mathcal{E}_{ij} + \dim \mathcal{E}_{ji} \leq 1$.
- 2°. If $\mathcal{E}_{ij}, \mathcal{E}_{jk}, \mathcal{E}_{ik} \neq 0$, then the multiplication $\mathcal{E}_{ij} \otimes \mathcal{E}_{jk} \rightarrow \mathcal{E}_{ik}$ is an isomorphism.

Proof. Suppose that $\mathcal{E}_{ij}^{\delta_1} \neq 0$ for some $\delta_1 > 0$. Let $A = S_i$ and $B = S_j[\delta_1]$. By Lemma 2.3, we have

$$B \in \left[\text{Ps}(A), \text{Ps}^{-1}(\tau(A[1])) \right].$$

Thus $\mathcal{E}_{ij}^m = 0$ for $m \neq \delta_1$ and $\mathcal{E}_{ji}^m = 0$ for $m > 1 - \delta_1$. But \mathcal{E}_{ji} is also concentrated in positive degrees and hence $\mathcal{E}_{ji} = 0$.

By Proposition 2.5, there is a quiver Q' such that, $\text{Ps}(A)$ consists precisely the projectives in $\text{mod } \mathbf{k}Q'$. Moreover, we have $B \in \text{mod } \mathbf{k}Q'$. Let $\mathbf{b} = \underline{\dim} B$ and $\mathbf{a} = \underline{\dim} A$, then we have

$$\begin{cases} \dim \text{Hom}(A, B) - \dim \text{Ext}^1(A, B) = \langle \mathbf{a}, \mathbf{b} \rangle = \dim \mathcal{E}_{ij}^{\delta_1}, \\ \dim \text{Hom}(B, A) - \dim \text{Ext}^1(B, A) = \langle \mathbf{b}, \mathbf{a} \rangle = \dim \mathcal{E}_{ji}^{\delta_1} = 0. \end{cases} \quad (3.4)$$

Since Q' is of Dynkin type, the quadratic form $q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$ is positive definite and, furthermore, since $A \not\cong B$, we have $\mathbf{a} \neq \mathbf{b}$. Hence

$$0 < \langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle = 2 - \langle \mathbf{a}, \mathbf{b} \rangle$$

i.e. $\dim \mathcal{E}_{ij}^{\delta_1} \leq 1$. Thus 1° follows.

For 2° , suppose that $\mathcal{E}_{jk}^{\delta_2} \neq 0$. Since $B \in \mathcal{H}'_Q$, Lemma 2.3 implies that

$$S_k[\delta_1 + \delta_2] \in (\mathcal{H}'_Q)[1] \cup \mathcal{H}'_Q.$$

Suppose that $\mathcal{E}_{ik}^{\delta_3} \neq 0$ and we have $C = S_k[\delta_3]$ is also in \mathcal{H}'_Q . Thus either $\delta_3 = \delta_1 + \delta_2$ or $\delta_3 = \delta_1 + \delta_2 - 1$.

Suppose that $\delta_3 = \delta_1 + \delta_2 - 1$. Let $\mathbf{c} = \underline{\dim} C$. As in (3.4), we have

$$\begin{cases} \langle \mathbf{a}, \mathbf{b} \rangle = 1, & \begin{cases} \langle \mathbf{a}, \mathbf{c} \rangle = 1, \\ \langle \mathbf{b}, \mathbf{a} \rangle = 0, \end{cases} & \begin{cases} \langle \mathbf{b}, \mathbf{c} \rangle = -1, \\ \langle \mathbf{c}, \mathbf{a} \rangle = 0, \end{cases} & \begin{cases} \langle \mathbf{c}, \mathbf{b} \rangle = 0. \end{cases} \end{cases}$$

Because A is simple, $\mathbf{a} \neq \mathbf{b} + \mathbf{c}$. But $\langle \mathbf{b} + \mathbf{c} - \mathbf{a}, \mathbf{b} + \mathbf{c} - \mathbf{a} \rangle = 0$, which is a contradiction. Therefore $\delta_3 = \delta_1 + \delta_2$.

Since A is a simple, any non-zero $f \in \text{Hom}(A, B)$ is injective and so gives a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow D \rightarrow 0$ in $\text{mod } \mathbf{k}Q'$. Applying $\text{Hom}(-, C)$ to it, we get an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(D, C) \rightarrow \text{Hom}(B, C) \xrightarrow{f^*} \text{Hom}(A, C) \rightarrow \\ \rightarrow \text{Hom}(D, C[1]) \rightarrow \text{Hom}(B, C[1]) = 0 \end{aligned}$$

If f^* is not an isomorphism, then $\text{Hom}(D, C) \neq 0$ and $\text{Hom}(D, C[1]) \neq 0$, contradicting Lemma 2.3. Hence multiplication $\mathcal{E}_{ij} \otimes \mathcal{E}_{jk} \rightarrow \mathcal{E}_{ik}$, i.e. composition $\text{Hom}(A, B) \otimes \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$, is an isomorphism, as required. \square

Lemma 3.4. *Let \mathcal{H} be a heart in $\mathcal{D}(Q)$ and S_i, S_j be two simples in $\text{Sim } \mathcal{H}$. Suppose that $\text{Hom}^1(S_i, S_j) = 0$. Let $\mathcal{H}_i = \mathcal{H}_{S_i}^\sharp, \mathcal{H}_j = \mathcal{H}_{S_j}^\sharp$ and $\mathcal{H}_{ij} = (\mathcal{H}_j)_{S_i}^\sharp$.*

1° . If $\text{Hom}^1(S_j, S_i) = 0$, then $(\mathcal{H}_i)_{S_j}^\sharp = \mathcal{H}_{ij}$.

2° . If $\text{Hom}^1(S_j, S_i) \neq 0$, let $T_j = \phi_{S_i}^{-1}(S_j)$. Then we have $\mathcal{H}_{ij} = (\mathcal{H}_)_{S_j}^\sharp$, where $\mathcal{H}_* = (\mathcal{H}_i)_{T_j}^\sharp$.*

$$\begin{array}{ccc}
& \mathcal{H}_i & \\
S_i \nearrow & & \searrow S_j \\
\mathcal{H} & & \mathcal{H}_{ij} \\
S_j \searrow & & \nearrow S_i \\
& \mathcal{H}_j &
\end{array}
\quad
\begin{array}{ccc}
& \mathcal{H}_i & \xrightarrow{T_j} \mathcal{H}_* \\
S_i \nearrow & & \searrow S_j \\
\mathcal{H} & & \mathcal{H}_{ij} \\
S_j \searrow & & \nearrow S_i \\
& \mathcal{H}_j &
\end{array}
\tag{3.5}$$

Proof. We have $\dim \operatorname{Hom}^\bullet(S_j, S_i) \leq 1$ by Lemma 3.3. Applying [35, Proposition 5.5], the lemma follows by a direct calculation. \square

Proposition 3.5. *If Q is of Dynkin type, then $\pi_1(\operatorname{EG}_N(Q, \mathcal{H}_Q))$ is generated by squares and pentagons as in (3.5) for any $N \geq 2$. Further, $\pi_1(\operatorname{EG}(Q))$ is generated by such squares and pentagons.*

Proof. Choose any cycle c in $\operatorname{EG}_N^\circ(Q, \mathcal{H}_Q)$. By Proposition 2.13,

$$B(c) = \{\mathcal{H} \mid \exists \mathcal{H}' \in c, \mathcal{H}' \leq \mathcal{H} \leq \mathcal{H}_Q[N-1]\}$$

is finite. We use induction on $\#B(c)$ to prove the first statement. If $\#B(c) = 1$, then c is trivial. Suppose that $\#B(c) > 1$ and any cycle $c' \subset \operatorname{EG}_N^\circ(Q, \mathcal{H}_Q)$ with $\#B(c') < \#B(c)$ is generated by the squares and pentagons. Choose a source \mathcal{H} in c such that $\mathcal{H}' \not\leq \mathcal{H}$ for any other source \mathcal{H}' in c . Let S_i and S_j be the arrows coming out at \mathcal{H} . If $i = j$ we can delete them in c to get a new cycle c' . If $i \neq j$, we know that $S_i : \mathcal{H} \rightarrow \mathcal{H}_i$ and $S_j : \mathcal{H} \rightarrow \mathcal{H}_j$ are either in a square or a pentagon as in (3.5). By the second part of [35, Lemma 5.8], we know that $\mathbf{H}_{N-1}(S_i) = 0$ and hence $\mathcal{H}_{ij} = (\mathcal{H}_j)_{S_i}^\# \in \operatorname{EG}_N^\circ(Q, \mathcal{H}_Q)$. Thus this square/pentagon are in $\operatorname{EG}_N^\circ(Q, \mathcal{H}_Q)$ and we can replace S_i and S_j in c by other edges in this square/pentagon to get a new cycle $c' \subset \operatorname{EG}_N^\circ(Q, \mathcal{H}_Q)$. Either way, we have $B(c') \subset (B(c) - \{\mathcal{H}\})$ for the new cycle c' and we are done.

Now choose any cycle c in $\operatorname{EG}(Q)$. By (2.11), we can choose $N \gg 1$ such that all hearts in $c[k]$ are in $\operatorname{EG}_N^\circ(Q, \mathcal{H}_Q)$ for some integer k . Then the second statement follows from the first one. \square

Lemma 3.6. *Any square or pentagon as in (3.5) is trivial in $\pi_1(\operatorname{Stab}(Q))$.*

Proof. Recall that we embed $\operatorname{EG}(Q)$ into $\operatorname{Stab}(Q)$. Suppose in case 2° of Lemma 3.4 and consider the path $L_p : \mathcal{H} \rightarrow \mathcal{H}_i \rightarrow \mathcal{H}_* \rightarrow \mathcal{H}_{ij}$ in $\operatorname{EG}(Q)$. Let $\operatorname{Sim} \mathcal{H} = \{S_1, \dots, S_n\}$.

Consider the stability condition σ whose heart is \mathcal{H} satisfying

$$\begin{cases}
Z(S_k) = \exp(\frac{1}{2}\pi\mathbf{i}) & k \neq i, j, \\
Z(S_i) = \exp(\delta\pi\mathbf{i}), \\
Z(S_j) = \exp(3\delta\pi\mathbf{i}),
\end{cases}$$

for some small $\delta > 0$. Notice that $\dim \operatorname{Hom}^\bullet(S_j, S_i) = 1$, hence there are only three indecomposables in \mathcal{H} generated by S_i and S_j , i.e. S_i, T_j and S_j , where T_j is the unique extension of S_j and S_i (with phase 2δ). Thus we can choose δ so small that any stable object other than S_i, T_j and S_j has phase larger than 4δ .

Consider the interval $L_0 = \{\sigma_\varepsilon\}_{\varepsilon \in (-4\delta, 0]}$, where $\sigma_\varepsilon = \varepsilon \cdot \sigma$. We have

$$\begin{cases} \sigma_\varepsilon \in \mathcal{U}(\mathcal{H}), & \varepsilon \in (-\delta, 0], \\ \sigma_\varepsilon \in \mathcal{U}(\mathcal{H}_i), & \varepsilon \in (-2\delta, -\delta), \\ \sigma_\varepsilon \in \mathcal{U}(\mathcal{H}_*), & \varepsilon \in (-3\delta, -2\delta), \\ \sigma_\varepsilon \in \mathcal{U}(\mathcal{H}_{ij}), & \varepsilon \in (-4\delta, -3\delta), \end{cases}$$

Therefore L_0 is homotopy to L_p . Notice that L_0 is contained in the contractible 'prism'

$$\mathbf{P} = \mathbb{C} \cdot \mathcal{U}(\mathcal{H}) \cong \mathbb{C} \cdot H^n,$$

where H is the upper half plane in (2.12). Similarly, the path $\mathcal{H} \rightarrow \mathcal{H}_j \rightarrow \mathcal{H}_{ij}$ is homotopy to some interval $L'_0 = \{\sigma'_\varepsilon\}_{\varepsilon \in (-4\delta', 0]}$ in \mathbf{P} , where σ' is the stability condition whose heart is \mathcal{H} satisfying

$$\begin{cases} Z'(S_k) = \exp(\frac{1}{2}\pi\mathbf{i}) & k \neq i, j, \\ Z'(S_i) = \exp(3\delta'\pi\mathbf{i}), \\ Z'(S_j) = \exp(\delta'\pi\mathbf{i}), \end{cases}$$

for some small $\delta' > 0$. Hence such pentagon is trivial. Same argument for the square. \square

Theorem 3.7. *If Q is of Dynkin type, then $\text{Stab}(Q)$ is simply connected.*

Proof. By Proposition 3.5 and Lemma 3.6 we know that $\pi_1(\text{EG}(Q))$ is trivial in $\text{Stab}(Q)$. Then the theorem follows from the surjection in Lemma 3.2. \square

4. SIMPLY CONNECTEDNESS OF CALABI-YAU DYNKIN CASE

4.1. The principal component. In this subsection, we show that $\text{EG}^\circ(\Gamma_N Q)$ induces a connected component in the space of stability conditions $\text{Stab}(\mathcal{D}(\Gamma_N Q))$.

Lemma 4.1. *$\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ is finite, for any heart $\mathcal{H} \in \text{EG}^\circ(\Gamma_N Q)$.*

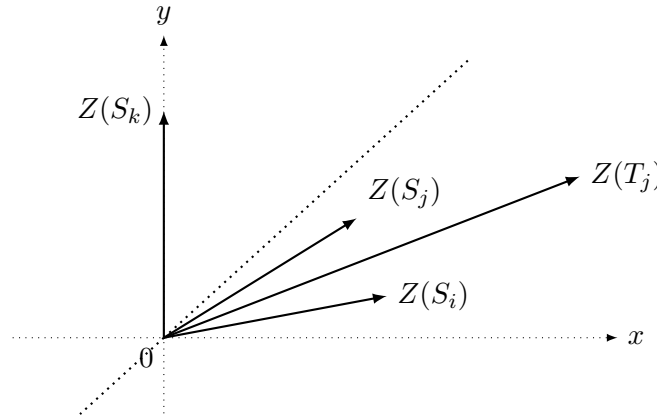


FIGURE 2.

Proof. By (2.10), we can assume that $\mathcal{H} \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ without lose of generality. By Theorem 2.12, we have isomorphism (2.9) and hence $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ is finite by Proposition 2.13.

Now we claim that, for $\mathcal{H} \in \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$, if $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_0)$ is finite for any $\mathcal{H}_Q[1] \leq \mathcal{H}_0 < \mathcal{H}$, then $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ is also finite.

If $\mathcal{H} \in \text{EG}_{N-1}^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$, then $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}) \subset \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$, which implies that $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ is finite. Now suppose that $\mathcal{H} \notin \text{EG}_{N-1}^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$. Let \mathcal{H} is induced from $\widehat{\mathcal{H}} \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ via \mathcal{I} , and we have $\widehat{\mathcal{H}} \notin \text{EG}_{N-1}^\circ(Q, \mathcal{H}_Q)$ by (2.9). By (2.5), for any simple $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$, there is some integer m such that $\widehat{S} \in \mathcal{H}_Q[m]$; and we have $1 \leq m \leq N-1$ by [35, Lemma 5.8]. Since $\widehat{\mathcal{H}} \notin \text{EG}_{N-1}^\circ(Q, \mathcal{H}_Q)$, there exists a simple $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$ such that $\mathbf{H}_{N-1}(\widehat{S}) \neq 0$, where \mathbf{H}_\bullet is with respect to \mathcal{H}_Q . By (2.5), $\widehat{S} \in \mathcal{H}_Q[N-1]$. Then $S = \mathcal{I}(\widehat{S}) \in \mathcal{H}_\Gamma[N-1]$. By [35, Lemma 5.8], we have

$$l(\mathcal{H}, S) \cap \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) = \{\mathcal{H}_S^{ib}\}_{i=0}^{N-2}.$$

By the inductive assumption, we know that $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^b)$ and $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^{(N-2)b})$ is finite; hence, so is

$$\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^\sharp) = \phi_S^{-1} \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^{(N-2)b}).$$

By [35, Lemma 8.1], we have

$$\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}) \subset \left(\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^b) \cup \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^\sharp) \right)$$

which implies the finiteness of $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$. Thus the lemma follows by induction. \square

Proposition 4.2. $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}) = \text{EG}_3(\Gamma_N Q, \mathcal{H})$, for any heart $\mathcal{H} \in \text{EG}^\circ(\Gamma_N Q)$.

Proof. Suppose that there exists a heart $\mathcal{H}' \in \text{EG}_3(\Gamma_N Q, \mathcal{H}) - \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$, we claim that there is an infinite directed path

$$\mathcal{H}_1 \xrightarrow{S_1} \mathcal{H}_2 \xrightarrow{S_2} \mathcal{H}_3 \rightarrow \dots$$

in $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ satisfying $\mathcal{H}_j < \mathcal{H}'$ for any $j \in \mathbb{N}$.

Use induction starting from $\mathcal{H}_1 = \mathcal{H}[1]$. Suppose we have $\mathcal{H}_j \in \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ such that $\mathcal{H}_j < \mathcal{H}'$. If for any simple $S \in \mathcal{H}_j$, we have $S \in \mathcal{H}'$, then $\mathcal{H}' \supset \mathcal{H}_j$ which implies $\mathcal{P}' \supset \mathcal{P}_j$, or $\mathcal{H}' \leq \mathcal{H}_j$; this contradicts to $\mathcal{H}_j < \mathcal{H}'$. Thus there is a simple $S_j \in \mathcal{H}_j$ such that $S_j \notin \mathcal{H}'$. Notice that $\mathcal{H}_j < \mathcal{H}' \leq \mathcal{H}[2] \leq \mathcal{H}_j[1]$, then by [35, Lemma 8.1], we have $\mathcal{H}_{j+1} = (\mathcal{H}_j)_{S_j}^\sharp \leq \mathcal{H}' (\leq \mathcal{H}[2])$. Notice that $\mathcal{H}' \notin \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$, therefore $\mathcal{H}_{j+1} \neq \mathcal{H}'$, which implies the claim.

Then we have that $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ is infinite, which contradicts to the finiteness in Lemma 4.1. \square

Similar to Section 3.1, we have the following results.

Theorem 4.3. We have the formula (3.2). Moreover, there is a principal component

$$\text{Stab}^\circ(\Gamma_N Q) = \bigcup_{\mathcal{H} \in \text{EG}^\circ(\Gamma_N Q)} \text{U}(\mathcal{H})$$

in $\text{Stab}(\mathcal{D}(\Gamma_N Q))$, which is the connected component containing $\text{U}(\mathcal{H}_\Gamma)$.

Proof. By Proposition 4.2, we have the finiteness condition $(\star\star)$ in [41, Section 2], and hence [41, Proposition 2.15 and Theorem 2.17] apply which implies the theorem. \square

We will also call a term in the RHS in (3.2) a face of the n -cell $U(\mathcal{H})$, for any $\mathcal{H} \in \text{EG}^\circ(\Gamma_N Q)$. Similarly, codimension one faces of \mathcal{H} corresponds to its simple tilts and, as in Section 3.1, we have the corresponding canonical embedding and surjection as below.

Proposition 4.4. *There is a canonical embedding (unique up to homotopy)*

$$\iota : \text{EG}^\circ(\Gamma_N Q) \hookrightarrow \text{Stab}^\circ(\Gamma_N Q) \quad (4.1)$$

such that the conditions 1° and 2° in Lemma 3.1. Moreover, we have a surjection $\pi_1(\text{EG}^\circ(Q)) \rightarrow \pi_1(\text{Stab}^\circ(Q))$.

4.2. Simply connectedness. Define the *basic cycles* in $\text{Stab}^\circ(\Gamma_N Q)/\text{Br}$ to be braid group orbits of lines in $\text{Stab}^\circ(\Gamma_N Q)$.

Theorem 4.5. *Suppose that Q is of Dynkin type and let $\mathcal{H} \in \text{EG}^\circ(\Gamma_N Q)$. Then*

$$\pi_1(\text{Stab}^\circ(\Gamma_N Q)/\text{Br}, [\mathcal{H}])$$

is generated by basic cycles containing $[\mathcal{H}]$ and it is a quotient group of the braid group Br_Q .

Proof. Let $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$, $\phi_k = \phi_{S_k}$ and let c_k be the basic cycle corresponding to $l(\mathcal{H}, S_k)$, for $k = 1, \dots, n$. Denote by p the quotient map

$$p : \text{Stab}^\circ(\Gamma_N Q) \rightarrow \text{Stab}^\circ(\Gamma_N Q)/\text{Br}.$$

We will drop $Y \in \{\text{Stab}^\circ(\Gamma_N Q)/\text{Br}, \text{Stab}^\circ(\Gamma_N Q)\}$ in $\pi_1(Y, y)$ if there is no ambiguity. By [11, Theorem 13.11], we have a short exact sequence

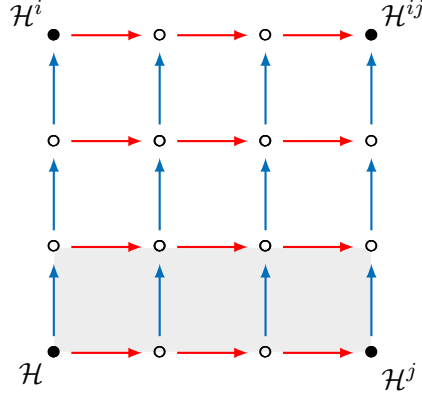
$$0 \longrightarrow p_*(\pi_1(\mathcal{H})) \longrightarrow \pi_1([\mathcal{H}]) \xrightarrow{\varrho} \text{Br}(\Gamma_N Q) \longrightarrow 0, \quad (4.2)$$

where ϱ sends c_k to ϕ_k^{-1} . What we only need to show is that $\{c_k\}$ satisfies the braid group relation and generates $\pi_1([\mathcal{H}])$.

For i, j satisfying $\text{Hom}^\bullet(S_i, S_j) = 0$, consider the lifting L_1 of $c_i c_j c_i^{-1} c_j^{-1}$ in $\pi_1(\mathcal{H})$ starting at \mathcal{H} . Let

$$\begin{aligned} \mathcal{H}^i &= \phi_i^{-1}(\mathcal{H}), & \mathcal{H}^{ji} &= \phi_j^{-1} \circ \phi_i^{-1}(\mathcal{H}), \\ \mathcal{H}^j &= \phi_j^{-1}(\mathcal{H}), & \mathcal{H}^{ij} &= \phi_i^{-1} \circ \phi_j^{-1}(\mathcal{H}). \end{aligned}$$

We have $\mathcal{H}^{ij} = \mathcal{H}^{ji}$ in this case. Then $L_1 \in \pi_1(\mathcal{H})$ is the boundary in Figure 3 with clockwise orientation. Notice that $\dim \text{Hom}^\bullet(S_j, S_i) \leq 1$ by Lemma 3.3. By the iterated use [35, Proposition 5.5] we can use $(N-1)^2$ squares, as in (3.5), to cover L_1 . For instance, Figure 3 is the CY-4 case, where the blue (resp. red) edges have direction S_i (resp. S_j) and the hearts are uniquely determined by these edges. Notice that using the same argument in Lemma 3.6, one can show any squares covering L_1 is trivial in $\pi_1(\mathcal{H})$. Thus L_1 is trivial in $\pi_1(\mathcal{H})$ which implies $c_i c_j = c_j c_i$ in $\pi_1([\mathcal{H}])$.

FIGURE 3. Square cover of L_1 , CY-4 case

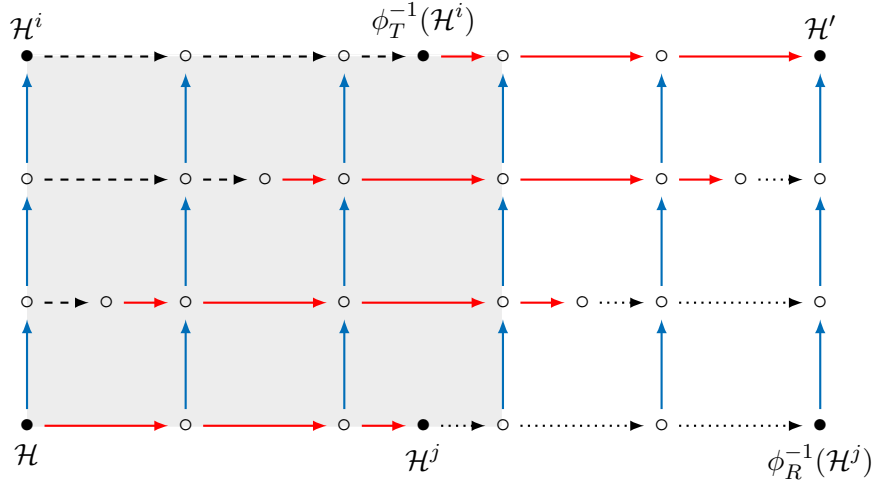
For i, j satisfying $\text{Hom}^\bullet(S_j, S_i) = \mathbf{k}[-1]$, consider the lifting L_2 of $c_i c_j c_i c_j^{-1} c_i^{-1} c_j^{-1}$ in $\pi_1(\mathcal{H})$ that starting at \mathcal{H} . Let $\mathcal{H}^i, \mathcal{H}^j$ as before and

$$\begin{aligned} T &= \phi_i^{-1}(S_j), R = \phi_j^{-1}(S_i), \\ \mathcal{H}' &= \phi_j^{-1} \circ \phi_T^{-1} \circ \phi_i^{-1}(\mathcal{H}). \end{aligned}$$

By [37, Lemma 2.11], we have

$$\phi_j^{-1} \circ \phi_T^{-1} \circ \phi_i^{-1} = \phi_i^{-1} \circ \phi_R^{-1} \circ \phi_j^{-1}.$$

Then $L_2 \in \pi_1(\mathcal{H})$ is the boundary in Figure 4 with clockwise orientation.

FIGURE 4. Square and pentagon cover of L_2 , CY-4 case

Similarly, we can use $(N-1)(2N-3)$ squares/pentagons, as in (3.5), to cover L_2 . For instance, Figure 4 is the CY-4 case, where the blue (resp. red, dashed, dotted)

edges have direction S_i (resp. S_j, T, R). Then we deduce that L_2 is trivial in $\pi_1(\mathcal{H})$ as before. Thus $c_i c_j c_i = c_j c_i c_j$ in $\pi_1([\mathcal{H}])$ as required.

To finish, we only need to show that $\{c_k\}_{k=1}^n$ generates $\phi_1([\mathcal{H}])$. By Theorem 2.12, we have $\text{EG}(\Gamma_N Q)/\text{Br} \cong \overline{\text{EG}}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ and hence $\pi_1(\text{EG}(\Gamma_N Q)/\text{Br})$ is generated by all squares and pentagons in $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ and basic cycles (c.f. [35]). These squares and pentagons are trivial as in Lemma 3.6. Therefore, it is essential to show that another basic cycle that does not contain $[\mathcal{H}]$ is generated by $\{c_k\}_{k=1}^n$.

Let $\mathcal{H}_i = \mathcal{H}_{S_i}^\sharp$, $T = \phi_i^{-1}(S_j)$, c_T be the basic cycle induced by the line $l(\mathcal{H}_i, T)$ and s_i be the path from \mathcal{H} to \mathcal{H}_i in the line $l(\mathcal{H}, S)$. Consider the basic cycle $s_i c_T s_i^{-1}$. If $\text{Hom}^\bullet(S_j, S_i) = 0$, let L_3 be the lifting of $(s_i c_T s_i^{-1}) c_i^{-1}$ in $\pi_1(\mathcal{H})$ stating at \mathcal{H} . As the gray area in Figure 3, we can cover L_3 using part of the covering for L_1 which implies $s_i c_T s_i^{-1} = c_i$. If $\text{Hom}^\bullet(S_j, S_i) = \mathbf{k}[-1]$, let L_4 be the lifting of $c_j (s_i c_T s_i^{-1}) c_j^{-1} c_i^{-1} \in \pi_1([\mathcal{H}])$ in $\pi_1(\mathcal{H})$ stating at \mathcal{H} . Similarly, we can cover L_4 using part of the covering for L_2 (as the gray area in Figure 4) which implies $s_i c_T s_i^{-1} = c_j^{-1} c_i c_j$. Either way, $s_i c_T s_i^{-1}$ is generated by $\{c_k\}_{k=1}^n$ as required. \square

Corollary 4.6. *Let Q be a Dynkin quiver. If the braid group action on $\mathcal{D}(\Gamma_N Q)$ is faithful, i.e. $\text{Br}(\Gamma_Q Q) \cong \text{Br}_Q$, then $\text{Stab}^\circ(\Gamma_N Q)$ is simply connected. In particular, this is true for Q of type A_n or $N = 2$.*

Proof. If $\text{Br}(\Gamma_Q Q) \cong \text{Br}_Q$, then by Theorem 4.5 and (4.2) we deduce that ϱ is an isomorphism. Hence $\pi_1(\text{Stab}^\circ(\Gamma_N Q)) = 1$ which implies the simply connectedness. The faithfulness for Q of type A_n follows from [37] and faithfulness for $N = 2$ follows from [5]. \square

Remark 4.7. By Theorem 4.5, basic cycles in $\text{Stab}^\circ(\Gamma_N Q)/\text{Br}$ are the generators of its fundamental group, which provide a topological realization of almost completed cluster tilting objects (c.f. [35, Remark 7.8]). In fact, our philosophy is that $\text{Stab}^\circ(\Gamma_N Q)/\text{Br}$ is the ‘complexification’ of the dual of cluster complex and provides the ‘right’ space of stability conditions for the higher cluster category

$$\mathcal{C}_{N-1}(Q) = \mathcal{D}(Q)/\Sigma_{N-1},$$

where $\Sigma_{N-1} = \tau^{-1} \circ [N-2] \in \text{Aut } \mathcal{D}(Q)$. Notice that there are no hearts in $\mathcal{C}_{N-1}(Q)$ and thus the space of stability conditions $\text{Stab}(\mathcal{C}_{N-1}(Q))$ is empty in the standard sense.

Here are two sensible conjectures.

Conjecture 4.8. *For any acyclic quiver Q , $\text{Br}(\Gamma_N Q) \cong \text{Br}_Q$.*

Conjecture 4.9. *For a Dynkin quiver Q , $\text{Stab}(\mathcal{D}(Q))$ and $\text{Stab}^\circ(\mathcal{D}(\Gamma_N Q))$ are contractible.*

5. A LIMIT FORMULA

In this section, we provide a limit formula for spaces of stability conditions.

Lemma 5.1. *If $\mathcal{H} = F_*(\hat{\mathcal{H}})$ for some heart $\hat{\mathcal{H}} \in \text{EG}^\circ(Q)$, then a stability condition $\hat{\sigma} = (\hat{Z}, \hat{\mathcal{P}})$ on $\mathcal{D}(Q)$ with heart $\hat{\mathcal{H}}$ canonically induces a stability condition $\sigma = (Z, \mathcal{P})$ with heart \mathcal{H} and such that $Z(F(\hat{S})) = \hat{Z}(\hat{S})$ for any $\hat{S} \in \text{Sim } \hat{\mathcal{H}}$. Thus we have a homomorphism $F_* : \text{U}(\hat{\mathcal{H}}) \rightarrow \text{U}(\mathcal{H})$.*

Proof. The heart $\widehat{\mathcal{H}}$ and \mathcal{H} are both good by Theorem 2.12. Thus the lemma follows by Proposition 2.16. \square

Theorem 5.2. *We have*

$$\text{Stab}(Q) \cong \lim_{N \rightarrow \infty} \text{Stab}^\circ(\Gamma_N Q) / \text{Br}(\Gamma_N Q)$$

in the following sense:

- 1°. *There exists a family of open subspaces $\{\mathcal{S}_N\}_{N \geq 2}$ in $\text{Stab}^\circ(Q)$ satisfying $\mathcal{S}_N \subset \mathcal{S}_{N+1}$ and $\text{Stab}(Q) \cong \lim_{N \rightarrow \infty} \mathcal{S}_N$.*
- 2°. *\mathcal{S}_N is homomorphic to a fundamental domain for $\text{Stab}^\circ(\Gamma_N Q) / \text{Br}$.*

Proof. Let $\text{Stab}_N^\circ(Q)$ and $\text{Stab}_N^\circ(\Gamma_N Q)$ be the interior of

$$\bigcup_{\mathcal{H} \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)} \overline{U(\mathcal{H})} \quad \text{and} \quad \bigcup_{\mathcal{H} \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)} \overline{U(\mathcal{H})}$$

respectively. By (3.2), we know that a face F_Q of some cell $U(\widehat{\mathcal{H}})$ is in $\text{Stab}_N^\circ(Q)$ if and only if $F_Q = U(\widehat{\mathcal{H}}) \cap U(\widehat{\mathcal{H}}')$ for some $\widehat{\mathcal{H}}, \widehat{\mathcal{H}}' \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ satisfying $\widehat{\mathcal{H}}[-1] \leq \widehat{\mathcal{H}}' < \widehat{\mathcal{H}}$. Similarly, a face F_Γ of some cell $U(\mathcal{H})$ is in $\text{Stab}_N^\circ(\Gamma_N Q)$ if and only if $F_\Gamma = \overline{U(\mathcal{H})} \cap U(\mathcal{H}')$ in $\text{Stab}_N^\circ(\Gamma_N Q)$ for some $\mathcal{H}, \mathcal{H}' \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ satisfying $\mathcal{H}[-1] \leq \mathcal{H}' < \mathcal{H}$. Notice that we have isomorphism (2.9) and formulae (3.2) (for $\text{Stab}_N^\circ(Q)$ as well as $\text{Stab}_N^\circ(\Gamma_N Q)$). Then by Lemma 5.1, we know that any such face F_Γ in $\text{Stab}_N^\circ(\Gamma_N Q)$ is induced from some face F_Q in $\text{Stab}_N^\circ(Q)$ via \mathcal{I} , in the sense that we have

$$\mathcal{I}_*(F_Q) = \mathcal{I}(\overline{U(\widehat{\mathcal{H}})} \cap U(\widehat{\mathcal{H}}')) = \overline{\mathcal{I}_*(U(\widehat{\mathcal{H}}))} \cap \mathcal{I}_*(U(\widehat{\mathcal{H}}')) = \overline{U(\mathcal{H})} \cap U(\mathcal{H}') = F_\Gamma$$

Thus we can glue the homomorphisms in Lemma 5.1 to a homomorphism

$$\mathcal{I}_* : \text{Stab}_N^\circ(Q) \rightarrow \text{Stab}_N^\circ(\Gamma_N Q).$$

Let $\mathcal{S}_N = \exp(-m\pi\mathbf{i}) \cdot \text{Stab}_N^\circ(Q)$, for $m = \lfloor -\frac{N}{2} \rfloor$, where \cdot is the \mathbb{C} -action. Then 1° follows from the limit formula in Proposition 2.13 and we have $\mathcal{S}_N \cong \text{Stab}_N^\circ(Q) \cong \text{Stab}_N^\circ(\Gamma_N Q)$, which completes the proof. \square

Example 5.3. The calculation of $\text{Stab}(A_2)$ and $\text{Stab}^\circ(\Gamma_N A_2)$ in [36, Section 7.5 and 7.6] (c.f. [1, p16, Figure 2]) illustrate the idea of the limit in Theorem 5.2 in A_2 case.

6. DIRECTED PATHS AND HN-STRATA

In this section, we will study the relations between directed paths in the exchange graph $\text{EG}(Q)$, HN-strata for \mathcal{H}_Q , slicings on $\mathcal{D}(Q)$ and stability functions on \mathcal{H}_Q .

6.1. Directed paths. Let $\text{EG}(Q; \mathcal{H}_1, \mathcal{H}_2)$ be the full subgraph of $\text{EG}(Q)$ consisting of hearts $\mathcal{H}_1 \leq \mathcal{H} \leq \mathcal{H}_2$. Denote by $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all directed paths from \mathcal{H}_1 to \mathcal{H}_2 in $\text{EG}(Q; \mathcal{H}_1, \mathcal{H}_2)$.

Lemma 6.1. *Suppose $\mathcal{H}_1 \leq \mathcal{H}_2$. Then $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2) \neq \emptyset$ if at least one of \mathcal{H}_1 and \mathcal{H}_2 is standard. In particular, we have*

$$\text{EG}(Q; \mathcal{H}[1], \mathcal{H}[N-1]) = \text{EG}_N(Q, \mathcal{H}) = \text{EG}_N^\circ(Q, \mathcal{H}),$$

for any standard heart $\mathcal{H} \in \text{EG}(Q)$.

Proof. Without lose of generality, suppose that $\mathcal{H}_1 = \mathcal{H}_Q[1]$ which is standard. For any simple $S_i \in \text{Sim } \mathcal{H}_2$, $S_i \in \mathcal{H}_Q[m_i]$ for some integer m_i by (2.5). Since $\mathcal{H}_1 \leq \mathcal{H}_2$, we have $m_i \geq 1$. Choose $N \gg 1$ such that $\mathcal{H}_2 \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ and then $\# \text{Ind}(\mathcal{P}_1 - \mathcal{P}_2)$ is finite. If $\mathcal{H}_1 < \mathcal{H}_2$, there exists j such that $m_j > 1$. By [35, Lemma 5.8], we can backward tilt \mathcal{H}_2 to $(\mathcal{H}_2)_{S_j}^\flat$ within $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$ which reduces $\# \text{Ind}(\mathcal{P}_1 - \mathcal{P}_2)$. Thus we can iterated backward tilt \mathcal{H}_2 to \mathcal{H}_1 by induction, which implies the lemma. \square

Define the directed distance $\text{dis}(\mathcal{H}_1, \mathcal{H}_2)$ and diameter $\text{diam}(\mathcal{H}_1, \mathcal{H}_2)$ between \mathcal{H}_1 and \mathcal{H}_2 to be the minimum and respectively maximum over the lengths of the paths in $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2)$. Recall the position function pf defined in Definition/Lemma 2.4. Since $\tau^{h_Q} = [-2]$, we have

$$\text{pf}(M[1]) - \text{pf}(M) = h_Q, \quad \forall M \in \Lambda(\mathcal{D}(Q)).$$

Here h_Q is the *Coxeter number*, which equals $n+1, 2(n-1), 12, 18, 30$ for Q of type A_n, D_n, E_6, E_7, E_8 respectively. There are the following easy estimations.

Lemma 6.2. *Suppose that $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2) \neq 0$. Let \mathcal{P}_i be the t -structure corresponding to \mathcal{H}_i . We have*

$$\text{diam}(\mathcal{H}_1, \mathcal{H}_2) \leq \# \text{Ind}(\mathcal{P}_1 - \mathcal{P}_2) \quad (6.1)$$

$$\text{diam}(\mathcal{H}_1, \mathcal{H}_2) \leq \# \text{Ind}(\mathcal{P}_2^\perp - \mathcal{P}_1^\perp) \quad (6.2)$$

$$\text{dis}(\mathcal{H}_1, \mathcal{H}_2) \geq \frac{\text{pf}(\mathcal{H}_2) - \text{pf}(\mathcal{H}_1)}{h_Q}. \quad (6.3)$$

In particular $\text{dis}(\mathcal{H}, \mathcal{H}[m]) \geq nm$ with equality for standard heart \mathcal{H} .

Proof. For any edge $\mathcal{H} \rightarrow \mathcal{H}_S^\sharp$, we have $\text{Ind } \mathcal{P} \supsetneq \text{Ind } \mathcal{P}_S^\sharp$ by Lemma 2.9 and hence (6.1) follows. Similarly for (6.2).

By [35, Proposition 5.5] we have formula [35, (5.8)]. Notice that T_j is a successor of S_j and hence $\text{pf}(T_j) > \text{pf}(S_j)$. We have

$$\text{pf}(\mathcal{H}_S^\sharp) - \text{pf}(\mathcal{H}) = \text{pf}(S[1]) - \text{pf}(S) + \sum_{j \in J_i^\sharp} (\text{pf}(T_j) - \text{pf}(S_j)) \geq \text{pf}(S[1]) - \text{pf}(S) = h_Q$$

which implies the inequality (6.3). In particular, if $\mathcal{H}_1 = \mathcal{H}, \mathcal{H}_2 = \mathcal{H}[m]$, the RHS of (6.3) equals mn .

Now suppose \mathcal{H} is standard, without loss of generality let $\mathcal{H} = \mathcal{H}_Q$. Label the simples S_1, \dots, S_n such that $\text{pf}(S_1) \leq \text{pf}(S_2) \leq \dots \leq \text{pf}(S_n)$. By Lemma 2.3, $\text{Hom}(M, L) \neq 0$ implies L is a successor of M and hence $\text{pf}(M) < \text{pf}(L)$. Thus $\text{Hom}^1(S_i, S_j) = 0$ for $i > j$. By [35, Proposition 5.5] we can tilt from \mathcal{H} to $\mathcal{H}[1]$ with respect to the simples S_n, \dots, S_1 in order, which implies $\text{dis}(\mathcal{H}, \mathcal{H}[m]) = mn$. \square

We can give a characterization of the longest paths in $\vec{\mathbf{P}}(\mathcal{H}_Q, \mathcal{H}_Q[1])$.

Proposition 6.3. *Let \mathcal{H} be a standard heart, then we have*

$$\text{diam}(\mathcal{H}, \mathcal{H}[1]) = \# \text{Ind } \mathcal{H}_Q = n \cdot \frac{h_Q}{2}. \quad (6.4)$$

Moreover, a path p in $\vec{\mathbf{P}}(\mathcal{H}, \mathcal{H}[1])$ has the longest length if and only if all vertices of p are standard hearts.

Proof. We can tilt from \mathcal{H} to $\mathcal{H}[1]$ by a sequence of APR-tiltings, which are L-tiltings. By Corollary A.2, such a path consisting of L-tiltings has length

$$\# \text{Ind}(\mathcal{P} - \mathcal{P}[1]) = \# \text{Ind}(\mathcal{P}[1]^\perp - \mathcal{P}^\perp) = \# \text{Ind} \mathcal{H}_Q.$$

Then the first claim follows from (6.1).

Suppose p is a longest path and use induction starting from \mathcal{H}_Q which is standard. Consider an edge $\mathcal{H} \rightarrow \mathcal{H}_S^\#$ in p with \mathcal{H} is standard. Since p is longest, by (6.1), we have

$$\# \text{Ind}(\mathcal{P} - \mathcal{P}_S^\#) = 1.$$

Notice that $S \in (\mathcal{P} - \mathcal{P}_S^\#)$, we have

$$\text{Ind} \mathcal{P}_S^\# = \text{Ind} \mathcal{P} - \{S\}.$$

Similarly, we have

$$\text{Ind}(\mathcal{P}_S^\#)^\perp = \text{Ind}(\mathcal{P})^\perp \cup \{S\}.$$

and hence

$$\text{Ind} \mathcal{P} \cup \text{Ind} \mathcal{P}^\perp = \text{Ind} \mathcal{P}_S^\# \cup \text{Ind}(\mathcal{P}_S^\#)^\perp. \quad (6.5)$$

By Proposition 2.5, the fact that a heart \mathcal{H}' is standard is equivalent to

$$\text{Ind} \mathcal{D}(Q) = \text{Ind} \mathcal{P}' \cup \text{Ind}(\mathcal{P}')^\perp.$$

Therefore, by (6.5), the standardness of \mathcal{H} implies the standardness of $\mathcal{H}_S^\#$. Thus the necessity follows.

On the other hand, if \mathcal{H} and its simple forward tilts $\mathcal{H}_S^\#$ are standard, we claim that it is an APR-tilting at a sink. Suppose not, that the vertex $V \in Q_0$ corresponding to S is not a sink. Then there is an edge $(V \rightarrow V') \in Q_1$ which corresponds to a nonzero map in $\text{Ext}^1(S, S')$, where S' is the simple corresponding to V' . Then $S \notin (\mathcal{P}_S^\#)^\perp$ since $S'[1] \in \mathcal{P}[1] \subset \mathcal{P}_S^\#$ by Lemma 2.9. Notice that $S \notin \mathcal{P}_S^\#$, we know that $\mathcal{H}_S^\#$ is not standard by Proposition 2.5, which is a contradiction. Thus if all the vertices of a path p are standard then it consisting of APR-tiltings, which are L-tiltings. By Corollary A.2, we know that the length of p is $\# \text{Ind} \mathcal{H}_Q$ which implies p is longest. \square

6.2. HN-strata. In this subsection, we use Reineke's notion of HN-strata to give an algebraic interpretation of

$$\vec{\mathbf{P}}(Q) := \vec{\mathbf{P}}(\mathcal{H}_Q, \mathcal{H}_Q[1]).$$

Definition 6.4. A (discrete) *HN-stratum* $[T_l, \dots, T_1]_{\text{HN}}$ in an abelian category \mathcal{C} is an ordered collection of objects T_l, \dots, T_1 in $\text{Ind} \mathcal{C}$, satisfying the HN-property:

- $\text{Hom}(T_i, T_j) = 0$ for $i > j$.

- For any nonzero object M in \mathcal{C} , there is an HN-filtration by short exact sequences

$$\begin{array}{ccccccc}
 0 = M_0 & \longrightarrow & M_1 & \longrightarrow & \dots & \longrightarrow & M_{m-1} \longrightarrow M_m = M \\
 & \nwarrow & \nearrow & & & & \nwarrow & \nearrow \\
 & & A_{j_1} & & & & A_{j_m} &
 \end{array} \tag{6.6}$$

with A_{j_i} is in $\langle T_{j_i} \rangle$ and $1 \leq j_m < \dots < j_1 \leq l$.

Notice that the uniqueness of HN-filtration follows from the first condition in HN-property. Denote by $\text{HN}(Q)$ the set of all HN-strata of \mathcal{H}_Q . We claim that there is a bijection between $\vec{\mathbf{P}}(Q)$ and $\text{HN}(Q)$.

Let $p = T_l \cdot \dots \cdot T_1$ be a path in $\vec{\mathbf{P}}(Q)$

$$p : \mathcal{H}_Q = \mathcal{H}_0 \xrightarrow{T_1} \mathcal{H}_1 \xrightarrow{T_2} \dots \xrightarrow{T_l} \mathcal{H}_l = \mathcal{H}_Q[1]$$

with corresponding t-structures $\mathcal{P}_0 \supset \mathcal{P}_1 \supset \dots \supset \mathcal{P}_l$. We have the following lemmas.

Lemma 6.5. *For any indecomposable M in \mathcal{H}_Q , there is a filtration as (6.6) such that A_{j_i} is in $\langle T_{j_i} \rangle$ and $1 \leq j_m < \dots < j_1 \leq l$.*

Proof. We construct such a filtration as follows. Since

$$M \in \mathcal{P}_0 - \mathcal{P}_l = \bigcup_{i=1}^l (\mathcal{P}_{i-1} - \mathcal{P}_i),$$

there exists an integer $0 < j \leq l$ such that $M \in \mathcal{P}_{j-1} - \mathcal{P}_j$. Since $\mathcal{H}_j = (\mathcal{H}_{j-1})_{T_j}^\sharp$, we have a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow A_j \longrightarrow 0$$

such that A_j in $\langle T_j \rangle$. This is the last short exact sequence in the required filtration. Since M' is in the torsion part corresponding to $(\mathcal{H}_{j-1})_{T_j}^\sharp$, we have

$$M' \in \mathcal{P}_j - \mathcal{P}_l = \bigcup_{i=j}^l (\mathcal{P}_{i-1} - \mathcal{P}_i).$$

Therefore we can repeat the procedure above and the lemma follows by induction. \square

Lemma 6.6. *Let $0 \leq j \leq l$. Let $\mathcal{F}_j = \langle T_1, \dots, T_j \rangle$ and $\mathcal{T}_j = \langle T_{j+1}, \dots, T_l \rangle$. Then $(\mathcal{F}_j, \mathcal{T}_j)$ is a torsion pair in \mathcal{H}_Q and $\mathcal{H}_j = (\mathcal{H}_Q)^\sharp$ with respect to this torsion pair.*

Proof. Use induction on j starting from the trivial case when $j = 0$. Now suppose that $\mathcal{H}_j = (\mathcal{H}_Q)^\sharp$ with respect to $(\mathcal{F}_j, \mathcal{T}_j)$. Since T_{j+1} is a simple in \mathcal{H}_{j+1} and $T_k \in \mathcal{P}_{j+1}$ for $k > j+1$, we have $\text{Hom}(T_k, T_{j+1}) = 0$ which implies $\text{Hom}(A, B) = 0$ for $A \in \mathcal{T}_{j+1}, B \in \mathcal{F}_{j+1}$. By Lemma 6.5 we know that for any object M in $\text{Ind } \mathcal{H}_Q$, there is a short exact sequence $0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0$ such that $A \in \mathcal{T}_{j+1}$ and $B \in \mathcal{F}_{j+1}$. Therefore $(\mathcal{F}_{j+1}, \mathcal{T}_{j+1})$ is a torsion pair in \mathcal{H}_Q . By Lemma 2.9, we have $\mathcal{H}_j \cap \mathcal{H}_Q = \mathcal{T}_j$. To finish we only need to show that $\mathcal{H}_{j+1} \cap \mathcal{H}_Q = \mathcal{T}_{j+1}$. This follows from $\mathcal{H}_{j+1} = (\mathcal{H}_j)_{T_j}^\sharp$. \square

Now we have an injection $\vec{\mathbf{P}}(Q) \rightarrow \text{HN}(Q)$ as follows.

Corollary 6.7. *Any directed path $p = p = T_l \cdot \dots \cdot T_1$ in $\vec{\mathbf{P}}(Q)$ induces an HN-stratum $[T_l, \dots, T_1]_{\text{HN}}$ in $\text{HN}(Q)$.*

Proof. Since $T_i \in \mathcal{F}_j$ and $T_j \in \mathcal{T}_j$ for $j > i$, $\text{Hom}(T_j, T_i) = 0$ follows from Lemma 6.6. Together with Lemma 6.5, the corollary follows. \square

For the converse construction, we have the following lemma.

Lemma 6.8. *Let $[T_l, \dots, T_1]_{\text{HN}}$ be an HN-stratum. For $0 \leq j \leq l$, let $\mathcal{F}_j = \langle T_1, \dots, T_j \rangle$ and $\mathcal{T}_j = \langle T_{j+1}, \dots, T_l \rangle$. Then $(\mathcal{F}_j, \mathcal{T}_j)$ is a torsion pair in \mathcal{H}_Q . Let $\mathcal{H}_j = (\mathcal{H}_Q)^\#$ with respect to this torsion pair. Then T_{j+1} is a simple in \mathcal{H}_j and $\mathcal{H}_{j+1} = (\mathcal{H}_j)^\#_{T_{j+1}}$.*

Proof. Similar to Lemma 6.6. \square

Combine the lemmas above, we have the following theorem.

Theorem 6.9. *The HN-stratas in $\text{HN}(Q)$ are precisely the directed paths in $\vec{\mathbf{P}}(Q)$.*

We will not distinguish $\text{HN}(Q)$ and $\vec{\mathbf{P}}(Q)$ from now on.

Corollary 6.10. *For any shortest path p in $\vec{\mathbf{P}}(Q)$, the set of labels of its edges are precisely $\text{Sim } \mathcal{H}_Q$.*

Proof. The HN-filtration of a simple in \mathcal{H}_Q (with respect to p) can only have one factor, i.e. itself. Hence any simple of \mathcal{H}_Q appears in an HN-stratum, and in particular, the labels of edges of p . Thus the length of p is at least n . By Lemma 6.2, the length of a shortest path p is exactly n and hence the corollary follows. \square

6.3. Slicing interpretation. We say a slicing \mathcal{S} of $\mathcal{D}(Q)$ is discrete if the abelian category $\mathcal{S}(\phi)$ is either zero or contains exactly one simple for any $\phi \in \mathbb{R}$. We say a heart \mathcal{H} is in a slicing \mathcal{S} if $\mathcal{H} = \mathcal{S}[\phi, \phi + 1)$ or $\mathcal{H} = \mathcal{S}(\phi, \phi + 1]$ for some $\phi \in \mathbb{R}$. Let $\text{Sli}^*(\mathcal{D}(Q), \mathcal{H})$ be the set of all discrete slicings of $\mathcal{D}(Q)$ that contain \mathcal{H} .

Definition 6.11. Let \mathcal{S}_1 and \mathcal{D}_2 in $\text{Sli}(\mathcal{D})$. If there is a monotonic function $\mathbb{R} \rightarrow \mathbb{R}$ such that $\mathcal{S}_1(\phi) = \mathcal{S}_2(f(\phi))$, then we say that the slicing \mathcal{S}_1 is homotopic (\sim) to \mathcal{S}_2 .

Now we can describe the relation between directed paths and slicings.

Proposition 6.12. *There is a canonical bijection $\text{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q)/\sim \rightarrow \text{HN}(Q)$.*

Proof. Let $\mathcal{S} \in \text{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q)$ and suppose $\mathcal{H}_Q = \mathcal{S}(I)$ for some interval I with $|I| = 1$. Then it induces an HN-stratum by taking the collection of objects which are simple in $\mathcal{S}(\phi)$ for $\phi \in I$ with decreasing order. On the other hand, an HN stratum $[K_l, \dots, K_1]_{\text{HN}}$ is induced by the slicing

$$\{\mathcal{P}(m + \frac{j}{l}) = \langle K_j[m] \rangle \mid j = 1, \dots, l, m \in \mathbb{Z}, \}.$$

Hence we have a surjection $\text{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q) \rightarrow \text{HN}(Q)$ while the condition that \mathcal{S}_1 and \mathcal{S}_2 maps to one HN-stratum is exactly the homotopy condition. \square

6.4. Total stability. Recall that we have the notion of a stability function on an abelian category (Definition 2.15). We call a stability function on \mathcal{A} *totally stable* if every indecomposable is stable. Reineke made the following conjecture.

Conjecture 6.13 ([39]). *Let Q be a Dynkin quiver. There exists a totally stable stability function on \mathcal{H}_Q*

This was first proved by Hille-Juteau (unpublished, see the comments after [20, Corollary 1.7]).

We say a stability condition on a triangulated category is *totally stable* if any indecomposable is stable. Let $\sigma = (Z, \mathcal{P})$ be a totally stable stability condition. Then it will induce a totally stable stability function Z on any abelian category $\mathcal{P}(I)$, for any half open half closed interval $I \subset \mathbb{R}$ with length 1; in particular, on its heart. On the other hand, a totally stable stability function on \mathcal{H}_Q will induce a stability condition on $\mathcal{D}(Q)$, which is also totally stable.

Now we give explicit examples to prove the existence of the totally stable stability condition on $\mathcal{D}(Q)$, which is a slightly weak version of Conjecture 6.13 because orientation matters.

Proposition 6.14. *Let Q be a Dynkin quiver. There exists a totally stable stability condition on $\mathcal{D}(Q)$.*

Proof. We treat the cases A, D and E separately.

For A_n -type, we use [38, Example A, Section 2]. Choose the orientation of Q as

$$n \longrightarrow n-1 \longrightarrow \cdots \longrightarrow 1$$

Let the stability function Z on \mathcal{H}_Q be defined by $Z(S_j) = -j + \mathbf{i}$. Then Z induces a totally stable stability condition on $\mathcal{D}(Q)$.

For D_n -type, choose the orientation of Q as

$$n-2 \longrightarrow n-3 \longrightarrow \cdots \longrightarrow 1 \begin{array}{l} \nwarrow^{n-1} \\ \swarrow_n \end{array}$$

Let the stability function Z on \mathcal{H}_Q be defined by

$$\begin{cases} Z(S_1) = \frac{n-3n}{2} + \mathbf{i}, \\ Z(S_j) = -j + \mathbf{i}, \quad j = 2, \dots, n-2, \\ Z(S_{n-1}) = Z(S_n) = \frac{6+3n-n^2}{4} + \mathbf{i}. \end{cases}$$

Notice that the τ -orbit of S_{n-2} in $\Lambda(\mathcal{H}_Q)$ is

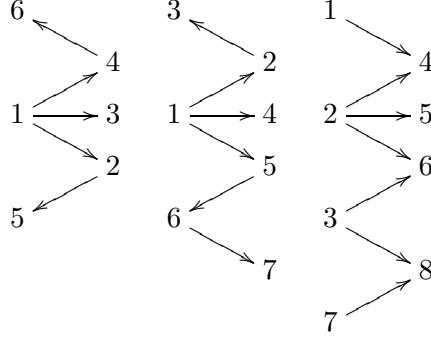
$$P_{n-2} - - M - - S_2 - - S_3 - - \cdots - - S_{n-2}$$

with central charges

$$\mathbf{i}, -1 + \mathbf{i}, -2 + \mathbf{i}, -3 + \mathbf{i}, \dots, -(n-2) + \mathbf{i}$$

it is easy to check that Z induces a totally stable stability condition on $\mathcal{D}(Q)$.

For the exceptional cases, we use Keller's quiver mutation program [23] to produce explicit examples of totally stable stability conditions for $E_{6,7,8}$. Choose the orientation of $E_{6,7,8}$ as follows



and we have following totally stable stability functions respectively:

$$\left\{ \begin{array}{l} Z(S_1) = 258 + 9\mathbf{i} \\ Z(S_2) = -53 + 32\mathbf{i} \\ Z(S_3) = -150 + 36\mathbf{i} \\ Z(S_4) = -75 + 33\mathbf{i} \\ Z(S_5) = -99 + 64\mathbf{i} \\ Z(S_6) = -101 + 10\mathbf{i} \end{array} \right. \left\{ \begin{array}{l} Z(S_1) = 165 + 10\mathbf{i} \\ Z(S_2) = -22 + 33\mathbf{i} \\ Z(S_3) = -35 + 36\mathbf{i} \\ Z(S_4) = -63 + 37\mathbf{i} \\ Z(S_5) = -14 + 28\mathbf{i} \\ Z(S_6) = -27 + 21\mathbf{i} \\ Z(S_7) = -39 + 24\mathbf{i} \end{array} \right. \left\{ \begin{array}{l} Z(S_1) = 47 + 16\mathbf{i} \\ Z(S_2) = 135 + 9\mathbf{i} \\ Z(S_3) = 93 + 11\mathbf{i} \\ Z(S_4) = -66 + 40\mathbf{i} \\ Z(S_5) = -57 + 32\mathbf{i} \\ Z(S_6) = -92 + 57\mathbf{i} \\ Z(S_7) = 42 + 25\mathbf{i} \\ Z(S_8) = -45 + 45\mathbf{i} \end{array} \right.$$

where S_i is the simple corresponding to vertex i . Figure 5 is the AR-quiver of the E_6 quiver under such a totally stable function, where the bullets are simples or origins, and the stars are other indecomposables.

□

6.5. Inducing directed paths. We call a stability function *discrete*, if μ_Z is injective when restricted to the stable indecomposables.

Proposition 6.15. [28] *Let $Z : \mathcal{K}(\mathcal{H}_Q) \rightarrow \mathbb{C}$ be a discrete stability function. Then the collection of its stable indecomposables in the order of decreasing phase is an HN-stratum of \mathcal{H}_Q .*

We say that a directed path in $\vec{\mathbf{P}}(Q)$ is *induced* if the corresponding HN-stratum is induced by some discrete stability function on \mathcal{H}_Q . Notice that, a totally stable stability function on \mathcal{H}_Q induced a directed path p_s in $\vec{\mathbf{P}}(Q)$ such that there is an edge M in p_s for any $M \in \text{Ind } \mathcal{H}_Q$. By (6.4), we know that p_s is the longest path in $\vec{\mathbf{P}}(Q)$. Thus, in the language of exchange graphs, Reineke's conjecture translates to, that there exists a longest path in $\vec{\mathbf{P}}(Q)$ which is induced.

It is natural to make a very strong generalization of Reineke's conjecture, that any path in $\vec{\mathbf{P}}(Q)$ is induced. However, this is not true, even for some longest path as below.

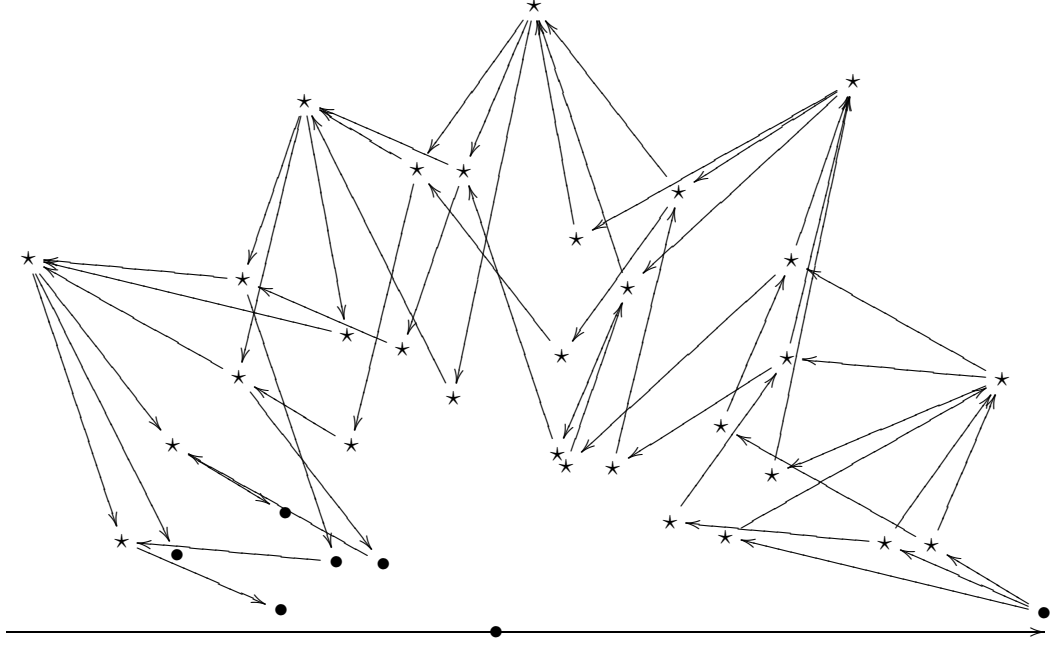
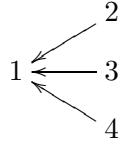
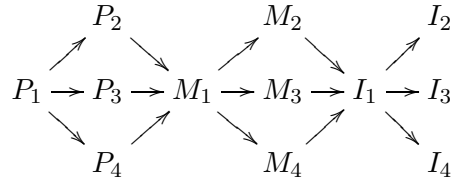


FIGURE 5. The AR-quiver of $\Lambda(\mathcal{H}_Q)$ E_7 -type under a totally stable stability function

Counterexample 6.16. Let Q be the following quiver of D_4 -type



Then the AR-quiver of \mathcal{H}_Q is

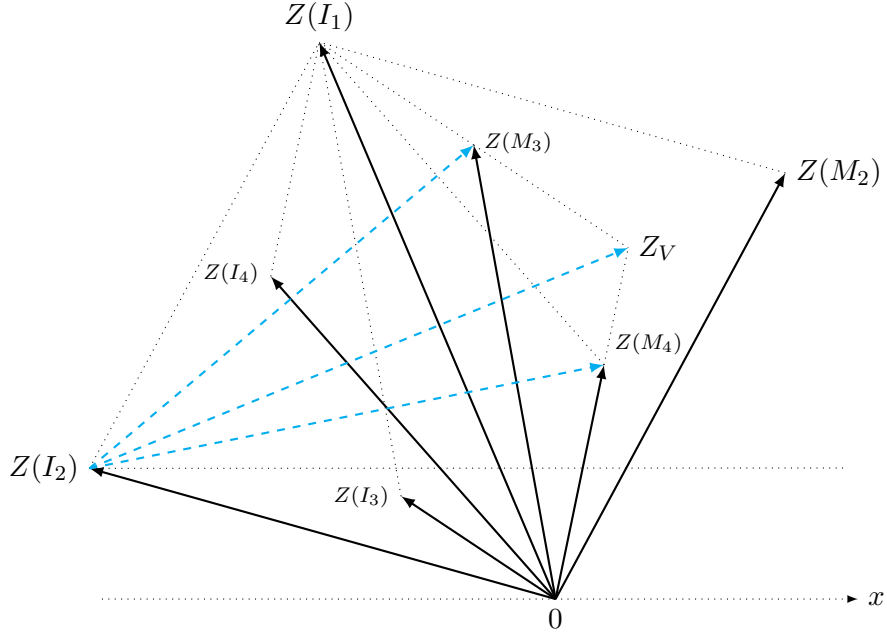


We claim that the following longest path

$$p = I_2 \cdot I_3 \cdot I_4 \cdot I_1 \cdot M_3 \cdot M_4 \cdot M_2 \cdot M_1 \cdot P_2 \cdot P_3 \cdot P_4 \cdot P_1 \quad (6.7)$$

is not induced. Suppose not, that p is induced by some stability function Z . The phase function μ_Z is decreasing on the edges in p from left to right in (6.7). Then $Z(I_3), Z(I_4), Z(M_3), Z(M_4)$ are in the parallelogram \mathfrak{P} with vertices $Z(I_2), Z(I_1), Z(M_2)$ and 0 , as shown in Figure 6. Let Z_V be the intersection of the line passing through points $Z(I_1), Z(M_3)$ and the line passing through points $Z(M_4), 0$. Notice that

$$\mu_Z(P_3), \mu_Z(P_4) \in [0, \mu_Z(I_2)],$$

FIGURE 6. The parallelogram \mathfrak{P}

we have

$$\begin{aligned}
 \mu_Z(P_3)\pi &= \arg(Z(M_4) - Z(I_2)) \\
 &< \arg(Z(M_4) - Z_V) \\
 &< \arg(Z(M_3) - Z(I_2)) \\
 &= \mu_Z(P_4)\pi,
 \end{aligned}$$

which is a contradiction.

This suggests another generalization of Reineke's conjecture as follows. We say two directed paths in $\vec{\mathbf{P}}(Q)$ are *weakly equivalent* if the unordered sets of their edges coincide.

Conjecture 6.17. *There is an induced path in each weak equivalence class in $\vec{\mathbf{P}}(Q)$.*

Notice that by (6.4), all longest paths in $\vec{\mathbf{P}}(Q)$ form a weak equivalent class E . Thus Reineke's conjecture can be stated as: there is a path in the weak equivalence class E which is induced.

7. QUANTUM DILOGARITHMS VIA EXCHANGE GRAPH

In this section, we define a DT-function on paths in exchange graphs, which provides another proof of Reineke's identities (see Theorem 7.1) and the existence of DT-type invariants for a Dynkin quiver.

7.1. DT-invariant for a Dynkin quiver. Let $q^{1/2}$ be an indeterminate and \mathbb{A}_Q be the quantum affine space

$$\mathbb{Q}(q^{1/2})\{y^\alpha \mid \alpha \in \mathbb{N}^{Q_0}, y^\alpha y^\beta = q^{\frac{1}{2}(\langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle)} y^{\alpha+\beta}\}, \quad (7.1)$$

where $\langle -, - \rangle$ is the Euler form associated to Q (see Section 2.1). Denote $y^{\dim M}$ by y^M for $M \in \mathcal{H}_Q$. Notice that \mathbb{A}_Q can be also written as

$$\mathbb{Q}(q^{1/2})\langle y^S \mid S \in \text{Sim } \mathcal{H}_Q \rangle / (y^{S_i} y^{S_j} - q^{\lambda_Q(i,j)} y^{S_j} y^{S_i}),$$

where $\lambda_Q(i, j) = \langle S_j, S_i \rangle - \langle S_i, S_j \rangle$. Let $\widehat{\mathbb{A}}_Q$ be the completion of \mathbb{A}_Q with respect to the ideal generated by $y^S, S \in \text{Sim } \mathcal{H}_Q$.

The DT-invariant $\text{DT}(Q)$ of the quiver Q can be calculated by the product (7.2) as follows.

Theorem 7.1 (Reineke [38], c.f. [20]). *For any HN-stratum $\varsigma = [K_1, \dots, K_l]_{\text{HN}}$ in $\text{HN}(Q)$, the product*

$$\text{DT}(Q; \varsigma) = \prod_{j=1}^l \mathbb{E}(y^{K_j}) \quad (7.2)$$

in \mathbb{A}_Q is actually independent of ς , where $\mathbb{E}(X)$ is the quantum dilogarithm defined as the formal series

$$\mathbb{E}(X) = \sum_{j=0}^{\infty} \frac{q^{j^2/2} X^j}{\prod_{k=0}^{j-1} (q^j - q^k)}.$$

In this subsection, we will review Reineke's approach to Theorem 7.1, via identities in the Hall algebra (closely following [20]).

Let \mathbf{k}_0 be a finite field with $q_0 = |\mathbf{k}_0|$ and consider $\mathcal{H}_Q(\mathbf{k}_0) = \text{mod } \mathbf{k}_0 Q$. Recall that the completed (non twisted, opposite) Hall algebra $\widehat{\mathbf{H}}_{\mathbf{k}_0}(Q)$ consists of formal series with rational coefficients

$$\sum_{[M] \in \mathcal{H}_Q} a_m[M],$$

where the sum is over all isomorphism classes $[M]$ in \mathcal{H}_Q . The product in $\widehat{\mathbf{H}}_{\mathbf{k}_0}(Q)$ is given by the formula

$$[L][M] = \sum c_{LM}^K(q_0)[K]$$

where $c_{LM}^K(q_0)$ is the number of submodules L' of K such that $L' \cong L$ and $K/L' \cong M$ in $\mathcal{H}_Q(\mathbf{k}_0)$. Then the HN-property of an HN-stratum $\varsigma = [K_1, \dots, K_l]_{\text{HN}}$ translates into the identity (in Hall algebra) as

$$\sum_{[M] \in \mathcal{H}_Q} [M] = \prod_{j=1}^l \sum_{[M] \in \langle K_j \rangle} [M] \quad (7.3)$$

Reineke showed that there is an algebra homomorphism (called *integration*)

$$\begin{aligned} \int : \widehat{\mathbf{H}}_{\mathbf{k}_0}(Q) &\rightarrow \widehat{\mathbb{A}}_{Q, q=q_0} \\ [M] &\mapsto q^{\langle \dim M, \dim M \rangle} \frac{y^M}{|\text{Aut } M|}. \end{aligned}$$

By integrating (7.3), a term $\sum_{[M] \in \langle K_j \rangle} [M]$ in the RHS gives $\mathbb{E}(y^{K_j})$ and hence the RHS gives $\text{DT}(Q; \varsigma)$. Notice that the LHS of (7.3) is clearly independent of ς , thus its integration gives the DT-invariant $\text{DT}(Q)$ for a Dynkin quiver Q .

Example 7.2. [20, Corollary 2.7] By the proof of Lemma 6.2, we know that $\overrightarrow{\prod}_{S \in \text{Sim } \mathcal{H}} S$ is a shortest path in $\overrightarrow{\mathbf{P}}(Q)$, where the product is with respect to the increasing order of the position function (if two objects have the same position function, then their order does not matter). Moreover, by direct checking, we know that $\overleftarrow{\prod}_{M \in \text{Ind } \mathcal{H}} M$ is a longest path in $\overrightarrow{\mathbf{P}}(Q)$ consisting of APR tiltings, where the product is with respect to the decreasing order of the position function. Then these two paths (or the corresponding HN-strata) give the equality

$$\overleftarrow{\prod}_{M \in \text{Ind } \mathcal{H}} \mathbb{E}(y^M) = \overrightarrow{\prod}_{S \in \text{Sim } \mathcal{H}} \mathbb{E}(y^S). \quad (7.4)$$

7.2. Generalized DT-invariants for a Dynkin quiver. We will give a combinatorial proof of Theorem 7.1, which provides a slightly stronger statement.

Let $p = \prod_{j=1}^l K_j^{\varepsilon_j} : \mathcal{H} \rightarrow \mathcal{H}'$ be a path (not necessarily directed) in $\text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1])$, where K_i are edges in $\text{EG}(Q)$ and the sign $\varepsilon_j = \pm 1$ indicates the direction of K_j in p . Define the DT-function of p to be

$$\text{DT}(Q; p) = \prod_{j=1}^l \mathbb{E}(y^{K_j})^{\varepsilon_j}.$$

Since we identify HN-strata with directed paths in Theorem 6.9, thus Theorem 7.1 can be rephrased as: the quantum dilogarithm of a directed path connecting \mathcal{H}_Q and $\mathcal{H}_Q[1]$ is independent of the choice of the path. It is natural to ask if the path-independence holds for more general paths (not necessary directed). The answer is yes within the subgraph $\text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1])$.

Theorem 7.3. *If p is a path in $\text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1])$, then $\text{DT}(Q; p)$ only depends on the head \mathcal{H} and tail \mathcal{H}' of p .*

Proof. We give a combinatorial proof. By Proposition 3.5, $\pi_1(\text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]))$ is generated by the squares and pentagons as in (3.5). Thus we only need to check these two cases for the path-independence.

Notice that in the square or pentagon, we have $\text{Hom}(S_i, S_j) = \text{Hom}(S_j, S_i) = 0$ and $S_i, S_j \in \mathcal{H}_Q$. In the square case we have

$$\text{Hom}^1(S_i, S_j) = \text{Hom}^1(S_j, S_i) = 0$$

and hence $\langle \dim S_i, \dim S_j \rangle = \langle \dim S_j, \dim S_i \rangle = 0$ by (2.2), which implies

$$y^{S_i} \cdot y^{S_j} = y^{S_j} \cdot y^{S_i}$$

and

$$\mathbb{E}(y^{S_i}) \cdot \mathbb{E}(y^{S_j}) = \mathbb{E}(y^{S_j}) \cdot \mathbb{E}(y^{S_i}) \quad (7.5)$$

as required. In the pentagon case we have a triangle $S_i \rightarrow T_j \rightarrow S_j \rightarrow S_i[1]$ and $\dim S_i + \dim S_j = \dim T_j$. Then

$$\text{Hom}^1(S_i, S_j) = 0, \quad \dim \text{Hom}^1(S_j, S_i) = 1$$

and hence $\langle \dim S_i, \dim S_j \rangle = 0$ and $\langle \dim S_j, \dim S_i \rangle = -1$ by (2.2). By the relations of the quantum affine space we have

$$\begin{aligned} y^{S_i} \cdot y^{S_j} &= q \cdot y^{S_j} \cdot y^{S_i}, \\ y^{T_j} &= q^{-\frac{1}{2}} \cdot y^{S_i} \cdot y^{S_j}. \end{aligned}$$

By the Pentagon Identity (see for example [20, Theorem 1.2]) we have

$$\mathbb{E}(y^{S_i}) \cdot \mathbb{E}(y^{S_j}) = \mathbb{E}(y^{S_j}) \cdot \mathbb{E}(y^{T_j}) \cdot \mathbb{E}(y^{S_i}) \quad (7.6)$$

as required. \square

Therefore for any two heart $\mathcal{H}_1, \mathcal{H}_2$ in $\text{EG}(Q; \mathcal{H}, \mathcal{H}[1])$, we have a *generalized DT-invariant*

$$\text{DT}(Q; \mathcal{H}_1, \mathcal{H}_2) := \text{DT}(Q; p) \quad (7.7)$$

where p is any path connecting \mathcal{H} and \mathcal{H}' . In particular, we have

$$\text{DT}(Q) = \text{DT}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]). \quad (7.8)$$

7.3. Wall crossing formula for APR-tilting. Let i be a sink in Q and $\text{Sim } \mathcal{H}_Q = \{S_j\}_{j=1}^n$. Then the APR-tilt $\mathcal{H}_{Q'} = (\mathcal{H}_Q)_{S_i}^\#$ is also a standard hearts in $D(Q)$, where Q' is obtained from Q by reversing the arrow at i . By [35, Proposition 5.5], we have $\text{Sim } \mathcal{H}_{Q'} = \{T_j\}_{j=1}^n$, where $T_i = S_i[1]$,

$$T_j = \text{Cone}(S_j \rightarrow S_i[1] \otimes \text{Ext}^1(S_j, S_i)^*)[-1]$$

for $j \neq i$. Let \dim' and $\langle -, - \rangle'$ be the dimension vector and the Euler form, respectively, associated to Q' . Consider the quantum affine space $\mathbb{A}_{Q'}$

$$\mathbb{Q}(q^{1/2}) \langle z^T \mid T \in \text{Sim } \mathcal{H}_{Q'} \rangle / (z^{T_i} z^{T_j} = q^{\lambda_{Q'}(i,j)} z^{T_j} z^{T_i})$$

where $z^S = z^{\dim' S}$ and $\lambda_{Q'}(i, j) = \langle T_j, T_i \rangle' - \langle T_i, T_j \rangle'$. By Theorem 7.3, we can also define DT-invariants $\text{DT}(Q'; \mathcal{H}_1, \mathcal{H}_2)$ in $\mathbb{A}_{Q'}$ for any $\mathcal{H}_1, \mathcal{H}_2 \in \text{EG}(Q; \mathcal{H}_{Q'}, \mathcal{H}_{Q'}[1])$.

Notice that the labels of edges in $\text{EG}(Q; \mathcal{H}_{Q'}, \mathcal{H}_Q[1])$ are in

$$\text{Ind}(\mathcal{H}_Q \cap \mathcal{H}_{Q'}) = \text{Ind } \mathcal{H}_Q - \{S_i\} = \text{Ind } \mathcal{H}_{Q'} - \{S_i[1]\}.$$

It is straightforward to check that the following conditions are equivalent

1°. for any hearts $\mathcal{H}_1, \mathcal{H}_2 \in \text{EG}(Q; \mathcal{H}_{Q'}, \mathcal{H}_Q[1])$,

$$\text{DT}(Q; \mathcal{H}_1, \mathcal{H}_2) = \text{DT}(Q'; \mathcal{H}_1, \mathcal{H}_2),$$

2°. we have $z^{T_i} = (y^{S_i})^{-1}$ and $z^M = y^M$ for any $M \in \text{Ind}(\mathcal{H}_Q \cap \mathcal{H}_{Q'})$.

3°. we have $z^{T_i} = (y^{S_i})^{-1}$ and $z^{T_j} = y^{T_j}$ for $j \neq i$.

4°. we have $z^{T_i} = (y^{S_i})^{-1}$ and $z^{S_j} = y^{S_j}$ for $j \neq i$.

Further, if the conditions above hold, the *wall crossing formula*

$$\text{DT}(Q) \cdot \mathbb{E}(y^{S_i})^{-1} = \mathbb{E}(y^{-S_i})^{-1} \cdot \text{DT}(Q') \quad (7.9)$$

comes for free because both sides equal to $\text{DT}(Q; \mathcal{H}_{Q'}, \mathcal{H}_Q[1])$.

Remark 7.4. One can rephrase Keller's green mutation formula (to calculate DT-invariants for quivers with potential) as DT-functions on the corresponding exchange graphs in the same way, while wall-crossing formula comes for free. In fact, exchange graphs are simplified (homological) version of Keller's cluster groupoids in [20], c.f. [35].

APPENDIX A. CONNECTEDNESS OF $\mathcal{D}(Q)$

We give two proofs of the connectedness of the exchange graph for $\mathcal{D}(Q)$, which was a result of Keller-Vossieck [25].

We say an indecomposable object L in a subcategory $\mathcal{C} \subset \mathcal{D}(Q)$ is *leftmost* if there is no path from any other indecomposable in \mathcal{C} to L , or equivalently that no predecessor of L is in \mathcal{C} . In particular, a leftmost object in a heart is simple. If in a simple forward tilting, the simple object is leftmost, we call it a *L-tilting*. Similarly, an indecomposable object R is *rightmost* if there is no path from any other indecomposable to L .

Lemma A.1. *Let S be leftmost in \mathcal{H} and $\mathcal{H}^\sharp = \mathcal{H}_S^\sharp$. We have*

1°. $(\text{Ind } \mathcal{H} \setminus \{S\}) \subset \mathcal{H}^\sharp$.

2°. Follow the notation of [35, Proposition 3.3]. If $m > 1$, then $H_m^\mathcal{F} = 0$.

3°. For any $M \in \text{Ind } \mathcal{D}(Q)$, $\text{Wid}_{\mathcal{H}^\sharp} M \leq \text{Wid}_{\mathcal{H}} M$.

Proof. Since S is a leftmost object, then $\text{Ind } \mathcal{F} = \{S\}$ and $\mathcal{F} = \{S^i \mid i \in \mathbb{Z}^+\}$. For any indecomposable in \mathcal{H} other than S , we have $\text{Hom}(M, S) = 0$ which implies $(\text{Ind } \mathcal{H} \setminus \{S\}) \subset \mathcal{T} \subset \mathcal{H}^\sharp$.

For 2°, suppose $H_m^\mathcal{F} = S^j \neq 0$, then $M[-k_m]$ is the predecessor of S . Consider an indecomposable summand L of H_1 . If $L = S$, then $S[k_1]$ is the predecessor of M . Since $k_1 > k_m$, S is the predecessor of $S[k_1 - k_m]$, hence the predecessor of $M[-k_m]$. Then M and S are predecessors to each other which is a contradiction. If $L \neq S$, then $L \in \mathcal{T}$. L is the predecessor of $M[-k_1]$, hence the predecessor of $M[-k_m]$. Then L is the predecessor of S which is also a contradiction.

For 3°, if $\text{Wid}_{\mathcal{H}} M > 0$, then $m > 1$. By 2°, $H_m^\mathcal{F} = 0$. Then by the filtration (3.2) of [35], $\text{Wid}_{\mathcal{H}^\sharp} M \leq k_1 - k_m = \text{Wid}_{\mathcal{H}} M$. If $\text{Wid}_{\mathcal{H}} M = 0$, or equivalently $m = 1$, then by the filtration (3.2) of [35] again, $\text{Wid}_{\mathcal{H}^\sharp} M = 0 = \text{Wid}_{\mathcal{H}} M$. \square

By the same argument in the proof of Lemma A.1 2°, we know that an object S is a leftmost object in a heart \mathcal{H} in $\mathcal{D}(Q)$, if and only if it is a leftmost object in the corresponding t-structure \mathcal{P} .

Corollary A.2. *For a L-tilting with respect to a leftmost object S , we have $\text{Ind } \mathcal{P}^\sharp = \text{Ind } \mathcal{P} - \{S\}$.*

Proof. Consider the filtration (3.1) of [35], we have $M \notin \mathcal{P}$ if and only if $k_m < 0$. If so, since $H_m^\mathcal{T}$ or $H_m^\mathcal{F}$ is not 0 in the filtration (3.1) of [35], then $M \notin \mathcal{P}^\sharp$. Thus $\text{Ind } \mathcal{P}^\sharp \subset \text{Ind } \mathcal{P}$. On the other hand, $M \in \text{Ind } \mathcal{P} - \text{Ind } \mathcal{P}^\sharp$ if and only if $H_m^\mathcal{F} \neq 0$ and $k_m = 0$. In which case, $m = 1$ by Lemma A.1, and hence $M = S$. \square

Lemma A.3. *For any object $M \in \text{Ind } \mathcal{D}(Q)$, if $\text{Wid}_{\mathcal{H}} M > 0$, then applying any sequence of L-tiltings to \mathcal{H} must reduce the width of M after finitely many steps.*

Proof. Suppose not, let $\text{Wid}_{\mathcal{H}} M > 0$ is the minimal width of M under any L-tilting. We have $m > 1$ in filtration (3.1) of [35]. Then $H_m^{\mathcal{F}} = 0$ by Lemma A.1. In the filtration (3.2) of [35], if $H_1^{\mathcal{T}}$ vanishes, then $\text{Wid}_{\mathcal{H}^\#} M \leq (k_1 - 1) - k_m < \text{Wid}_{\mathcal{H}} M$. But $\text{Wid}_{\mathcal{H}} M$ is minimal, thus $H_1^{\mathcal{T}} \neq 0$ after any L-tilting.

Consider the size of $H_1^{\mathcal{T}}$. Let $H_1^{\mathcal{T}} = \bigoplus_{j=1}^l T_j^{s_j}$, where T_j are different indecomposables in \mathcal{T} and l is a positive integer. By the filtration (3.2) of [35] we know $H_1^{\mathcal{T}}$ will not change if we do L-tilting that is not with respect to any T_j . And if we do L-tilting with respect to some T_j , then $H_1^{\mathcal{T}}$ will lose the summand T_j . Since $H_1^{\mathcal{T}}$ can not vanish, we can assume after many L-tilting, l reaches the minimum, and we can not do L-tilting that is with respect to any T_i .

On the other hand, for any object $M \in \text{Ind } \mathcal{D}(Q)$, while $M[m]$ is the successor of some T_j when m is large enough, it can not be leftmost in any heart that contains T_j . Besides we can only do L-tilting with respect to any object once. Thus, we will eventually have to tilt T_i , which will reduce l and it is a contradiction. \square

Now we have a proposition about how one can do L-tilting.

Proposition A.4. *Applying any sequence of L-tiltings to any heart, will make it standard after finitely many steps.*

Proof. By Lemma A.3, the width of any particular indecomposable will become zero after finitely steps in the sequence. But, up to shift, there are only finitely many indecomposables in $\text{Ind } \mathcal{D}(Q)$. Thus, after finitely steps, we must reach a heart with respect to which all indecomposables have width zero and which is therefore standard, by Proposition 2.5. \square

Now we can prove the connectedness:

Theorem A.5 (Keller-Vossieck [25]). *$\text{EG}(Q)$ is connected.*

Proof. Since t-structure and heart are one-one correspondent, any heart is connected to a standard heart by Proposition A.4. On the other hand, using the equivalent definition 3° in Proposition 2.5 for ‘standard’, the set of all standard hearts is connected by APR-tilting (c.f. [2, page 201]). So the theorem follows. \square

APPENDIX B. STABILITY SPACE OF $\mathcal{D}(A_2)$

Let Q be the quiver of type A_2 with orientation $2 \rightarrow 1$ and $\text{Ind } \mathcal{H}_Q = \{C_1, C_2, C_3\}$ such that $\text{Ext}^1(C_3, C_1) \neq 0$. Write C_{3m+i} for $C_i[m]$. We have $\text{Aut } \mathcal{D}(A_2) \simeq \mathbb{Z}\langle \xi \rangle$, where the generator $\xi = \tau \circ [1]$ satisfying $\xi(C_j) = C_{j+1}$ and $\xi^3 = [1]$.

Lemma B.1. *Let $\sigma = (Z, \mathcal{P})$ be a stability condition in $\mathcal{D}(A_2)$. There exists an element $\zeta \in \text{Aut } \mathcal{D}(A_2)$ and an nonnegative integer m such that the simples in the heart of $\zeta \circ \sigma$ are C_1 and $C_3[m]$. In particular, there are three types of stability conditions on $\mathcal{D}(A_2)$:*

- *Every indecomposable object is stable.*
- *Up to shift, two indecomposables are stable and one is semistable (but not stable).*
- *Up to shift, two indecomposables are stable and one is not semistable.*

Proof. Notice that $\text{EG}(Q)$ is connected. By [35, Proposition 5.5], we know the changes of simple during tilting. Then The first assertion follows by direct calculating. By comparing the phases of C_1 and C_3 with respect to the stability condition $\xi \circ \sigma$, we get the three cases. \square

Let

$$\begin{aligned}\tilde{U} &= \{(Z, \mathcal{P}) \in \text{Stab}(A_2) \mid C_j \text{ are stable for } j = 1, 2, 3\}, \\ \tilde{W}_j &= \{(Z, \mathcal{P}) \in \text{Stab}(A_2) \mid C_j \text{ is not semistable}\}, \quad j = 1, 2, 3.\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}\partial \tilde{W}_j &= \{(Z, \mathcal{P}) \in \text{Stab}(A_2) \mid C_j \text{ is semistable but not stable}\}, \\ \partial \tilde{U} &= \partial \tilde{W}_1 \cup \partial \tilde{W}_2 \cup \partial \tilde{W}_3, \\ \text{Stab}(A_2) &= \tilde{U} \cup \partial \tilde{U} \cup \tilde{W}_1 \cup \tilde{W}_2 \cup \tilde{W}_3.\end{aligned}$$

Notice that the intersection of \mathbb{C} -actions and $\text{Aut } \mathcal{D}(A_2)$ is \mathbb{Z} with generator $-1 \in \mathbb{C}$ or $[1] \in \text{Aut } \mathcal{D}(A_2)$. Therefore we have a commutative diagram:

$$\begin{array}{ccc} & \text{Stab}(A_2) & \\ \swarrow / \text{Aut} & & \searrow / \mathbb{C} \\ \mathcal{M}_A & & \mathcal{M}_C \\ \searrow / \mathbb{C}^* & & \swarrow / c_3 \\ & \mathcal{M} & \end{array} \tag{B.1}$$

where $\mathcal{C}_3 = \text{Aut } \mathbb{Z}[1], \mathbb{C}^* = \mathbb{C}/\mathbb{Z}$ and $\mathcal{M} = \text{Aut } \mathcal{D}(A_2) \setminus \text{Stab}(A_2)/\mathbb{C}$. Let $U, W_j \subset \mathcal{M}_C$ be the quotient spaces of \tilde{U} and \tilde{W}_j in \mathcal{M}_C respectively. We have a conformal isomorphism $f : R \rightarrow \overline{W_2} \cup U$ (see Figure 7), where

$$R = \{\Theta = x + y\mathbf{i} \mid x < 1\} \subset \mathbb{C} \tag{B.2}$$

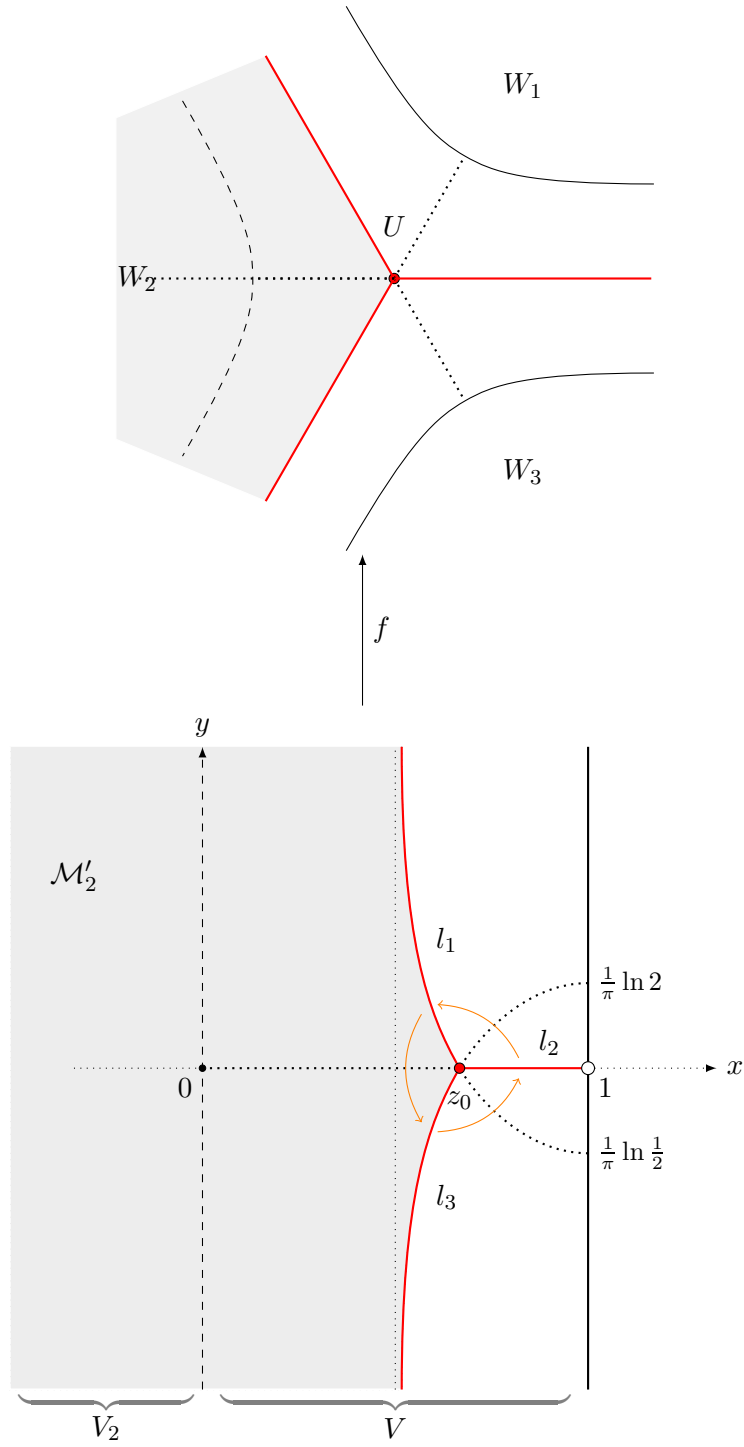
such that $f(\Theta) = [\sigma]$ in \mathcal{M}_C and the stability condition $\sigma = (Z, \mathcal{P})$ is determined by the following conditions:

- $Z(C_1) = 1$ and $Z(C_3) = \exp(i\pi\Theta)$;
- The simples in the heart of σ are C_1 and $C_3[m]$, where $m = -[\text{Im } \Theta]$.

Let $V = f^{-1}(U)$ and $V_2 = f^{-1}(W_2)$. Denote \mathbb{T} the triangle with vertices $T_1 = 1, T_2 = 0$ and $T_3 = -Z(C_3)$. The \mathcal{C}_3 -action on U will identify the stability conditions whose corresponding triangles \mathbb{T} are similar to each other. The red lines l_i in Figure 7 correspond to the case when \mathbb{T} is an isosceles triangle (with vertex angle at T_i), where

$$\begin{cases} l_1 = \{\Theta = x + y\mathbf{i} \mid x \in (\frac{1}{2}, \frac{2}{3}], y\pi = -\ln(-2\cos x\pi)\}, \\ l_2 = \{\Theta = x + y\mathbf{i} \mid y = 0, x \in [\frac{2}{3}, 1)\}, \\ l_3 = \{\Theta = x + y\mathbf{i} \mid x \in (\frac{1}{2}, \frac{2}{3}], y\pi = \ln(-2\cos x\pi)\}.\end{cases}$$

Moreover let $\omega_0 : \mathcal{M}_C \rightarrow \mathcal{M}_C$ be the conformal map with order 3 corresponding to the \mathcal{C}_3 -action and sending W_j to W_{j+1} . Also denote by ω_0 , the induced \mathcal{C}_3 -action on V . Denote \mathcal{M}'_2 the region strictly right bounded by $l_1 \cup l_3$ in Figure 7. Then $\mathcal{M}_2 = f(\mathcal{M}'_2)$ is a fundamental domain for the quotient map $\mathcal{M}_C \rightarrow \mathcal{M}$.


 FIGURE 7. The conformal isomorphism $f : R \rightarrow \overline{W_2} \cup U$

Lemma B.2. \mathcal{M} can be obtained from $\overline{\mathcal{M}'_2}$ by identifying the points on the boundary $l_1 \cup l_3$ with respect to the reflection of x -axis, Moreover, $z_0 = l_1 \cap l_2 \cap l_3 = \frac{2}{3}$ is the only orbitfold point in $\partial\mathcal{M}$, which is with order $\frac{1}{3}$.

Proof. The lemma follows from the facts that $\omega_0(l_j) = l_{j+1}$ and $l_1 \cap l_2 \cap l_3 = \{0\}$. \square

Let $\mathcal{M}_3 = \omega_0(\mathcal{M}_2)$ and $\mathcal{M}_1 = \omega_0(\mathcal{M}_3)$. By Lemma B.2, we have

$$\mathcal{M}_C = \overline{\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3} \quad \text{and} \quad \overline{\mathcal{M}_{j-1}} \cap \overline{\mathcal{M}_j} = f(l_{j+1}).$$

Lemma B.3. We have a conformal isomorphism $g : \mathcal{M}_C \xrightarrow{\cong} \mathbb{C}$.

Proof. Let $l(j) = \{z \in \mathbb{C} \mid \arg z = \frac{2\pi}{3}j\}$. Using Riemann mapping theorem and Reflection Principle (as in [33, Lemma 4.4]), we have a map g'_2 sending $\overline{\mathcal{M}'_2}$ conformally isomorphic to

$$\mathcal{M}(2) = \{z \in \mathbb{C} \mid \arg z \in [\frac{2\pi}{3}, \frac{4\pi}{3}]\},$$

such that $g'_2(\bar{z}) = \overline{g'_2(z)}$. Let $g_2 = g'_2 \circ f^{-1}$, then we have $g_2 : \mathcal{M}_2 \xrightarrow{\cong} \mathcal{M}(2)$. Define $\omega : \mathbb{C} \rightarrow \mathbb{C}$ by $\omega(z) = z \cdot \exp(\frac{2\pi}{3}\mathbf{i})$ and let

$$\mathcal{M}(3) = \omega(\mathcal{M}(2)), \quad \mathcal{M}(1) = \omega(\mathcal{M}(3)).$$

Then we have two conformal isomorphisms

$$\begin{aligned} g_1(z) &= \omega^{-1} \circ g_2 \circ \omega_0 : \mathcal{M}_1 \xrightarrow{\cong} \mathcal{M}(1), \\ g_3(z) &= \omega \circ g_2 \circ \omega_0^{-1} : \mathcal{M}_3 \xrightarrow{\cong} \mathcal{M}(3). \end{aligned}$$

By [10, Theorem 11-8], we can conformally extend g_j to the smooth boundary

$$f(l_{j-1} \cup l_{j+1} - \{z_0\})$$

such that $g \circ f(l_{j\pm 1}) = l(j \pm 1)$. Notice that the extended maps g_1, g_2 and g_3 agree on

$$f(l_1 \cup l_2 \cup l_3 - \{z_0\})$$

by a direct calculation, thus we obtain a conformally isomorphism

$$g : \mathcal{M}_C - \{f(z_0)\} \longrightarrow \mathbb{C} - \{0\}.$$

Then by [10, Theorem 11-8] again, we can conformally extend g to the boundary $\{f(z_0)\}$ which implies the lemma. \square

Theorem B.4. $\text{Stab}(A_2)$ is isomorphic to \mathbb{C}^2 as complex manifold.

Proof. The theorem follows from $\text{Stab}(A_2)/\mathbb{C} \simeq \mathcal{M}_C \simeq \mathbb{C}$ and $H^1(\mathbb{C}, \mathcal{O}) = 0$. \square

APPENDIX C. STABILITY SPACE OF CALABI-YAU-N A2-TYPE

C.1. Autequivalences and the universal cover. Let S_1, S_2 be the simples in the standard heart \mathcal{H}_Γ in $\mathcal{D}(\Gamma_N A_2)$ such that $\text{Ext}^1(S_1, S_2) \neq 0$. Then the braid group $\text{Br}(\Gamma_N A_2) \cong \text{Br}_3$ has a set of generators ϕ_{S_1}, ϕ_{S_2} . By [9], $\xi^3 = [3N - 4]$ generates of the center of Br_3 , where $\xi = \phi_{S_2}^{-1} \circ \phi_{S_1}^{-1}$. Let $\text{Aut}_0(\Gamma_N A_2)$ be the subgroup of $\text{Aut } \mathcal{D}(\Gamma_N A_2)$

which is generated by ϕ_{S_1}, ϕ_{S_2} and [1]. By [35, Proposition 8.8], we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}[3N-4] & \longrightarrow & \mathrm{Br}(\Gamma_N A_2) & \longrightarrow & P_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathbb{Z}[1] & \longrightarrow & \mathrm{Aut}_0(\Gamma_N A_2) & \longrightarrow & P_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}_{3N-4} & \xlongequal{\quad} & \mathbb{Z}_{3N-4} & &
 \end{array} \tag{C.1}$$

where $\mathrm{Br}(\Gamma_N A_2) = \mathrm{Br}_3$, $P_2 = \mathrm{PSL}_2(\mathbb{Z})$, and hence $\mathrm{Aut}_0(\Gamma_N A_2) \cong \mathrm{Br}_3$. Therefore we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathrm{Stab}^\circ(\Gamma_N A_2) & & \\
 & \swarrow \scriptstyle / \mathrm{Br}(\Gamma_N A_2) & & \searrow \scriptstyle / \mathbb{C} & \\
 \mathcal{L}_B^N & \xrightarrow{\scriptstyle / \mathbb{Z}[3N-4]} & \mathcal{L}_A^N & & \mathcal{L}_C^N \\
 & \searrow \scriptstyle / \mathbb{C}_B^* & \swarrow \scriptstyle / \mathbb{C}_A^* & \searrow \scriptstyle / P_2 & \\
 & & \mathcal{L}^N & &
 \end{array} \tag{C.2}$$

Moreover, let $\Delta = \{\alpha_1, \alpha_2, \alpha_3 \mid \alpha_1 = \alpha_2, \alpha_2 = \alpha_3 \text{ or } \alpha_3 = \alpha_1\}$, $W = \mathfrak{S}_3$ and $\Delta_0 = \Delta/W$. We have (c.f. [7])

$$\begin{aligned}
 \mathfrak{h}^{reg} &= \left\{ f(x) = \prod_{j=1}^3 (x - \alpha_j) \mid \sum_{j=1}^3 \alpha_j = 0, \alpha_j \in \mathbb{C} \right\} \backslash \Delta \\
 &\quad \downarrow W \\
 \mathfrak{h}^{reg}/W &= \{f(x) = x^3 - a \cdot x + b \mid a, b \in \mathbb{C}\} \backslash \Delta_0
 \end{aligned} \tag{C.3}$$

Write $\mathfrak{Q} = \mathfrak{h}^{reg}/W$ and denote by C^U the universal cover of \mathfrak{Q} . Thus we have the following commutative diagram:

$$\begin{array}{ccc}
 & C^U & \\
 \pi_0 \swarrow & & \searrow \scriptstyle / \mathbb{C} \\
 \mathfrak{Q} & & H \\
 \searrow \scriptstyle / \mathbb{C}^* & & \swarrow \scriptstyle / P_2 \\
 & J &
 \end{array} \tag{C.4}$$

where $J = H/P_2$ is the j -line. Recall that H is the upper half plane in \mathbb{C} and the j -line is an orbitfold surface with two orbitfold points (of orders 2 and 3).

If $N = 2$, we can identify (see [8]) (C.4) with the right square of (C.2). We will show that this identification works for $N > 2$ in the following subsection.

C.2. Deformations. Let $N \geq 2$. Let $\mathcal{N}|_t$ be the area right bounded by $l_1 \cup l_3$ and left bounded by $b_t = \{x = -t\}$ (see Figure 8). We have the following lemma.

Lemma C.1. *The orbitfold \mathcal{L}^N can be obtained from $\mathcal{N}|_{(N-2)/2}$ by gluing its boundary $l_1 \cup l_3 \cup b_{(N-2)/2}$ with respect to the reflection of x -axis.*

Proof. Recall that $\text{Sim } \mathcal{H}_\Gamma = \{S_1, S_2\}$ with $\text{Ext}^1(S_2, S_1) \neq 0$. By Lemma 5.1, we have a conformal map

$$\alpha : V \cup \mathcal{N}|_{(N-2)/2} \rightarrow \mathcal{L}^N$$

sending Θ to $[\sigma]$, where the stability condition $\sigma = (Z, \mathcal{P})$ is determined by the following conditions

- $Z(S_1) = 1$ and $Z(S_2) = \exp(i\pi\Theta)$;
- The simples in the heart of σ are S_1 and $S_3[m]$, where $m = -\lfloor \text{Im } \Theta \rfloor$.

The surjectivity of α follows by Theorem 2.12. To complete the proof, it is essential to show that for $\Theta_1 \neq \Theta_2 \in V \cup \mathcal{N}|_{(N-2)/2}$ satisfying $\alpha(\Theta_1) = \alpha(\Theta_2)$, we have

- 1°. either $\Theta_1, \Theta_2 \in b_{(N-2)/2}$ such that $\Theta_1 + \Theta_2 = 2 - N$.
- 2°. or $\Theta_1, \Theta_2 \in V$ such that $\omega_0^k(\Theta_1) = \Theta_2$, where $k \in \{\pm 1\}$ and ω_0 is the \mathcal{C}_3 -action on V sending l_i to l_{i+1} .

Let σ_1 and σ_2 be the corresponding stability conditions. Notice that for any σ in the orbit of $\alpha(z)$, if $z \in \mathcal{N}|_0$, then there are three (up to shift) indecomposables are semistable; otherwise there are two. Therefore $\alpha(\Theta_1) = \alpha(\Theta_2)$ implies Θ_i are both in $\mathcal{N}|_0$ or neither.

Suppose that $\Theta_1, \Theta_2 \in \mathcal{N}|_{(N-2)/2} - V$. Notice that the two stable objects (up to shift) are S_1 and S_2 . Consider the central charges and phases of them with respect to σ_i . Since $\text{Ext}^1(S_2, S_1) = \text{Ext}^1(S_1, S_2[N-2])$, either we have

$$\begin{cases} \frac{Z_1(S_1)}{Z_1(S_2)} = \frac{Z_2(S_1)}{Z_2(S_2)}, \\ \varphi_1(S_1) - \varphi_1(S_2) = \varphi_2(S_1) - \varphi_2(S_2). \end{cases} \quad (\text{C.5})$$

or

$$\begin{cases} \frac{Z_1(S_1)}{Z_1(S_2)} = \frac{Z_2(S_2[N-2])}{Z_2(S_1)}, \\ \varphi_1(S_1) - \varphi_1(S_2) = \varphi_2(S_2[N-2]) - \varphi_2(S_1). \end{cases} \quad (\text{C.6})$$

where ϕ_i is the phase function with respect to σ_i , for $i = 1, 2$. Equation (C.5) implies $\sigma_1 = \sigma_2$ which is a contradiction. Hence equation (C.6) holds, which implies $\Theta_1 + \Theta_2 = 2 - N$ as required in 1°.

Now let $\Theta_1, \Theta_2 \in U$. Then up to shift, there are three semistable objects S_1, S_2 and $\phi_{S_2}(S_1)$. Consider their central charges and we know that the triangles \mathbb{T}_1 and \mathbb{T}_2 are similar, where \mathbb{T}_i has vertices $0, Z_i(S_1)$ and $Z_i(S_2)$. This condition exactly means that σ_i differs by a \mathcal{C}_3 -action (c.f. Section 5) as required in 2°. \square

Lemma C.2. *We have a conformal isomorphism $\mathcal{L}^N \cong \mathcal{L}^2$ for any $N > 1$.*

Proof. Let $X(N) = g(\mathcal{N}|_{(N-2)/2})$ where g is the map in Lemma B.3. We only need to prove that $X(N)$ is conformally isomorphic to $X(2)$. Consider $Y(N) = \bigcup_{j=1}^3 \omega^j X(N)$. By Riemann mapping theorem, there is a conformal isomorphism $h : Y(N) \rightarrow Y(2)$ such that $h(0) = 0$ and $h'(0) = 1$ (see Figure 9). Let $h_j = \omega^{-j} \circ h \circ \omega^j$ for any $j \in \mathbb{Z}$. Since

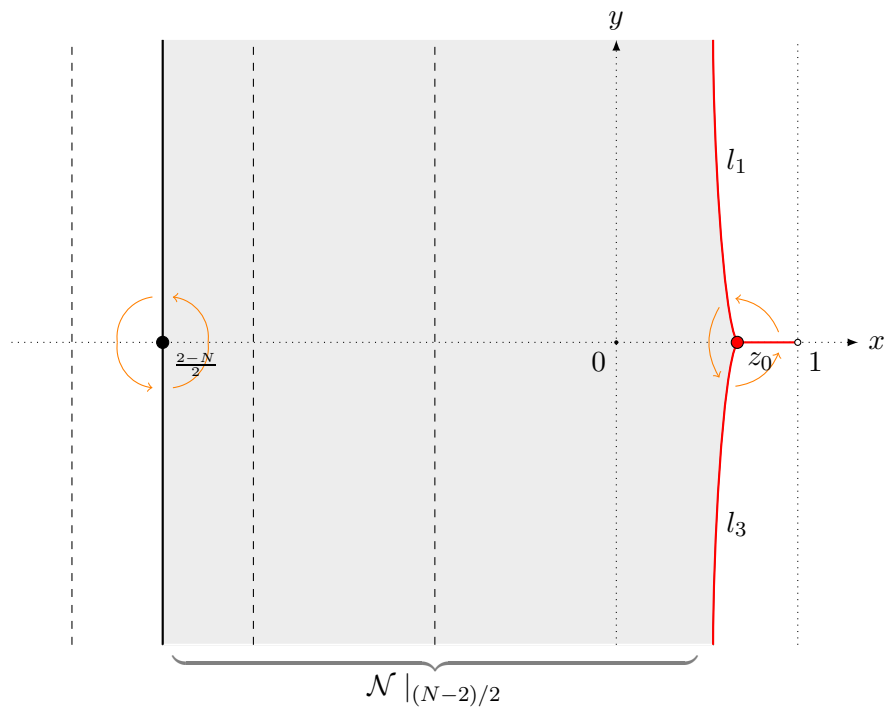
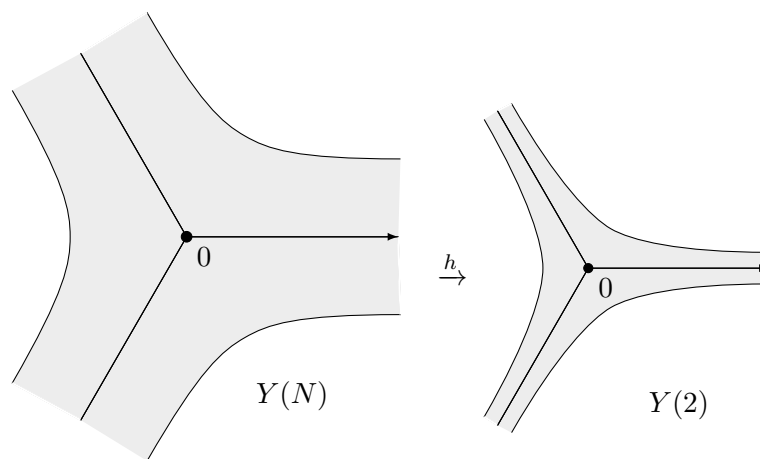

 FIGURE 8. j -line for $\text{Stab}^\circ(\Gamma_N A_2) / \text{Aut}_0(\Gamma_N A_2)$


FIGURE 9. Deformation

$h_j(0) = 0$ and $h'_j(0) = 1$, we have $h = h_j$ by the uniqueness of Riemann mapping theorem. Notice that $Y(N)$ and $Y(2)$ are symmetry with respect to x -axis by construction, hence $h(l(0)) = l(0)$ by Reflection Principle. Then $h = h_j$ implies $h(l(j)) = l(j)$ for any $j \in \mathbb{Z}$ and hence $h|_{X(N)}: X(N) \rightarrow X(2)$ is a conformal isomorphism as required. \square

Theorem C.3. *We have $\text{Stab}^\circ(\Gamma_N A_2) \cong \text{Stab}^\circ(\Gamma_2 A_2) \cong C^U$ as complex manifold.*

Proof. By Lemma C.2, we have $\mathcal{L}^N \cong \mathcal{L}^2 \cong J$. Since the \mathbb{C}_A^* -bundle \mathcal{L}_A^N is the principal bundle over \mathcal{L}^N , we have $\mathcal{L}_A^N \cong \mathcal{L}_A^2 \cong \mathfrak{Q}$. Finally, we have $\text{Aut}_0(\Gamma_N A_2) \cong \text{Aut}_0(\Gamma_x A_2) \cong \text{Br}_3$. Hence $\text{Stab}^\circ(\Gamma_N A_2)$ and $\text{Stab}^\circ(\Gamma_2 A_2)$ are both the universal cover of \mathfrak{Q} which implies the assertion. \square

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